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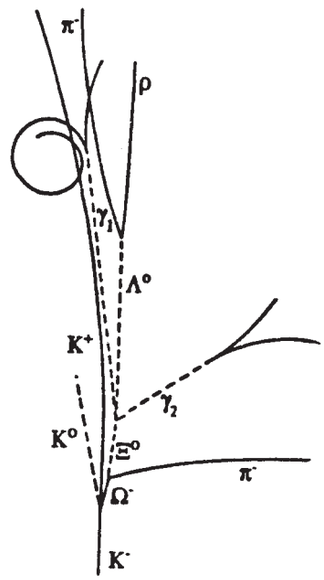

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DIMENSIONLESS PHYSICAL CONSTANT MYSTERIES (I) THE FINE STRUCTURE CONSTANT

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Abstract

Feynman proposed searching for $-\alpha^{1/2} = -0.08542455$ with the $\pm$ sign on the $\alpha^{1/2}$ for the *positive* and *negative* charge, and may be related to π, e, 2 and 5. We found $\alpha^{1/2} \approx \pm\frac{\log e}{\Phi\pi} = \pm 0.0854372$ where $\Phi = \frac{1}{\phi} = 2\cos(\frac{\pi}{5})$. I/FQHE $R_{xy} = \pm Z_0/2\nu_i\alpha$ unveils $\alpha^{1/2} = \log e^{\pm\phi/K\pi}$ where $\Phi - \phi - e - \pi$ in Euler Identity and $K \sim \{3, 37, 61\}$ from $2^{(\mathbf{p}-1)}(\mathbf{p}-1)! \in 2^n n!$ are linked to Quantum theories. The electroweak unification at high-energy level $\alpha^{1/2}(m_Z^2) = \pm\frac{\log F}{\Phi\pi}(1 - \alpha \cdot \sin^2\theta_w)$, where Fransén-Robinson constant $F = \int_0^\infty \frac{dt}{\Gamma(t)} = 2.80777\ldots$ replaced $e = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n)} = 2.71828\ldots$

Keywords: Fine structure constant, Quantum Theory, I/FQHE, Magnetic monopole.

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1 Brief History of the α Math Formulas

1-1 Introduction of the Fine Structure Constant

The elementary charge of electron $\mathbf{e}$ was proposed by Stoney in 1894 and discovered by Thomson in 1896,[1,2] then Planck introduced the energy quanta $h\nu$ in 1901 and explained as photon $\epsilon = h\nu = hc/\lambda$ by Einstein in 1905.[3,4]

Planck first noticed in 1905 that $\mathbf{e}^2/c$ and h have the same dimension.[5] In 1909, Einstein found that there are two fundamental velocities in physics: c and $\mathbf{e}^2/h$ requiring explanation. He said, *"It seems to me that we can conclude from $h = \mathbf{e}^2/c$ that the same modification of theory that contains the elementary quantum $\mathbf{e}$ as a consequence, will also contain as a consequence* ***the quantum structure of radiation.***"[6] Later on, Einstein noticed that "three decimals are missing." He prompted $hc/\mathbf{e}^2 \sim 879$ (i.e., $\alpha^{-1} = 140(5)$), which was subsequently rejected by Lorentz since "900 seems too much."[6] His guesswork was before Millikan's oil-drop experiment to measure the elementary charge $\mathbf{e}$ and Avogadro Constant N_A.[7]

In 1913, Jeans suggested $h/2\pi = (4\pi\mathbf{e})^2/c$, i.e., $\alpha^{-1} = hc/2\pi e^2 = (4\pi)^2 = 157.91367$.[8] In 1914, two American electrochemists, Lewis and Adams proposed that "all universal constants involve only integral numbers and π", and presented [9]

$$h = \frac{\mathbf{e}'^2}{c}\left(\frac{8\pi^5}{15}\right)^{1/3} = \frac{4\,(4\pi\mathbf{e})^2}{c}\left(\frac{\pi^5}{5!}\right)^{1/3} \tag{1.1}$$

where $\mathbf{e}' = 4\pi\mathbf{e}$. From (1.1)

$$\alpha^{-1} = \frac{\hbar c}{\mathbf{e}^2} = 32\pi\left(\frac{\pi^5}{5!}\right)^{1/3} = 137.348 \tag{1.2}$$

which is derived from Stefan-Boltzmann law $\varepsilon_T = a\mathrm{T}^4$. They noticed that the dimension of the radiation constant $a = k^4(\varepsilon/\theta^4)$ is $(energy \times length)^{-3}$ while $\mathbf{e}'^2$ is $(energy \times length)$. $8\pi^5/15$ in (1) can be obtained from the dimensionless integration in 3D blackbody radiation

$$8\pi \cdot J = 8\pi \int_0^\infty \frac{x^3 dx}{e^x - 1} = 8\pi\Gamma(4)\zeta(4) = \frac{8\pi^5}{15} = \frac{4^3\pi^5}{5!} \tag{1.3}$$

where *zeta*-function $\zeta(4)$ was solved by Euler using Wallis products. Lewis applied a cube root to (1.3) since it involves a 3D volume. In 1915, Allen rewrote (1.1) as $\alpha = (15/\pi^2)^{1/3}/(4\pi)^2$.[10] Therefore, the study of an α math formula came before Sommerfeld's fine structure constant formula of relativistic discrete H-spectra in 1916, which gives the experimental value $\alpha^2 = 5.30 \times 10^{-5}$ (i.e., $\alpha^{-1} = 137.360563948$ closed to (1.2)). Sommerfeld was the first to pin-down the approximate value of α and also point out that α serves as a bridge in the quantum h to electricity **e**, relativity c.[11]

1-2 Early Exploration and Wyler's formula

Since (1.1) involves blackbody radiation, people believed that α may be linked to temperature. In 1931, $T_0 = -(2/\alpha - 1) = -273$ [K] was erroneously published in *Naturwissenschaften*.[12] The Lewis–Adams conjecture was discussed among physicists. In 1935, Heisenberg wrote to Dirac: "*I do not believe at all any more in your conjecture that the Sommerfeld fine-structure constant may have something to do with the concept of temperature; that is, neither do I any more believe in the Lewis value*". Indeed, Lewis' value is wrong, but his idea lead to another dimensionless constant $\alpha_R = \frac{2}{\pi}\left(\frac{\pi^5}{5!}\right)^{1/3}\alpha = \left[\frac{\zeta(4)}{\zeta(2)}\right]^{1/3}\alpha = \frac{1}{157.555}$, involving the continued spectra of blackbody radiation.[13] Heisenberg wrote to Bohr with a joke formula suggested by Lunn in 1922

$$\alpha^{-1} = 2^4 3^3/\pi = 137.5\dot{0}987 \tag{1.4}$$

Bohr replied with the golden angle $(2\pi/\Phi^2)(180/\pi) = 360\phi^2 = 137.5077643^\circ$ for studying phyllotaxis in 1909

$$\alpha^{-1} = 360/\Phi^2 = 137.5077643 \tag{1.5}$$

Pauli worked hard in 1935, and once suggested that five-dimensional relativity theory might help to understand the problem.[14]

Following Pauli, Wyler's formula in 1969 exposed a similar pattern with the Lewis formula (1.2), but in fourth root and in the reciprocal way [15]

$$\begin{aligned} \alpha &= \frac{9}{16\pi^3}\left(\frac{\pi}{5!}\right)^{1/4} = \frac{9}{16\pi^4}\left(\frac{\pi^5}{5!}\right)^{1/4} = \left(\frac{3}{4\pi^2}\right)^2\left(\frac{\pi^5}{5!}\right)^{1/4} \\ &= 1/137.0360824 = 0.00729734813 \end{aligned} \tag{1.6}$$

Wyler used Hua's result in "Harmonic analysis of functions of several complex variables in the classical domains",[16] without discussing the physical dimensional analysis, and proposed a new assembling formula according to the 5D Dirac relativistic equation

$$V(D_n) = \frac{\pi^n}{2^{n-1}n!}; V(S_{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}; \qquad V(C_n) = \frac{\pi 2\pi^{n/2}}{\Gamma(n/2)} \leftarrow Hua \tag{1.7}$$

$$\alpha = \frac{8\pi \cdot [V(D_n)]^{1/(n-1)}}{V(S_{n-1})V(C_n)} \leftarrow Wyler$$

let $n = 5$ in (1.7), then

$$V(D_5) = \frac{\pi^5}{2^4 5!}; \quad V(S_{5-1}) = \frac{8\pi^2}{3}; \quad V(C_5) = \frac{8\pi^3}{3} \leftarrow Hua \tag{1.8}$$

$$\alpha = \frac{8\pi(\pi^5/2^4 5!)^{1/4}}{(8\pi^2/3)(8\pi^3/3)} = \frac{9}{16\pi^3}\left(\frac{\pi}{5!}\right)^{1/4} \leftarrow Wyler$$

The puzzling Wyler formula in (1.6) has many variations, such as

$$\alpha^{-1} = \sqrt[4]{\frac{1^0 \cdot 4^0 \cdot 5^1 \cdot 8^0 \cdot 9^0}{2^{19} \cdot 3^7 \cdot 6^0 \cdot 7^0 \cdot 0^0}\pi^{11}} = 2^{-19/4} \cdot 3^{-7/4} \cdot 5^{1/4} \cdot \pi^{11/4} \tag{1.9}$$

He also reported $\beta = m_p/m_e = 3 \times 2\pi^5 = 6!(\pi^5/5!) = 1836.1181$. Unfortunately, he failed to provide an origin for his ideas or an explanation of their use. In 1925, Rice suggested $2\pi\alpha^{-1} = hc/e'^2 = (\frac{8\pi^3}{3})(R\rho/r^2)$.[17] Born suggested $\alpha^3 = (\frac{2}{\pi})(\frac{8\pi^2}{3})(e^2\nu/m_e c^3)$ and tested $\alpha = \frac{e^2}{\hbar c} = \frac{2\pi\gamma}{1.236} = \Phi\pi\gamma$ in 1935,

and proposed the reciprocity principle in 1939.[18] In 1951, Lenz suggested that $m_p/m_e = 6\pi^5$.[19] Wyler's formula came after these works. The above history shows that this line of study is a combination of the n-space dimensionless volume and blackbody radiation with Pauli's idea. Obviously, 3D (x-y-z), 4D (x-y-z-t) or 5D (x-y-z-t-E) is based on the experimental data *137*. In 1989, Bailey and Ferguson used a supercomputer to check Wyler's formula, and automatically produced several "other relations of comparable complexity with even better accuracy." One example is $\alpha^{-5} = 150\pi(6^5 5^2 \pi^3)^8$ (i.e., $\alpha^{-1} = 137.036048362143$).[20] This clearly showed that a Wyler-type formula could not be the *unique* answer for the fine structure constant. Wyler's formula is later discussed in the $E8$ lie groups. In 2006, Castro reviewed the coupling constant with the Complex Domains.[21]

1-3 More Debating Formulas

Wyler's work made people devise simpler ways to obtain the magic number, with no more care given to physical dimensional analysis. Here are some popular examples:

A Wyler-type formula in 1971[22]

$$\alpha^{-1} = 2^{5/3} \cdot 3^{-8/3} \cdot 5^{5/2} \cdot \pi^{7/3} = 137.036006449 \tag{1.10}$$

A formula with 4π in 1971[22]

$$\alpha^{-1} = \pi + \pi^2 + 4\pi^3 = \pi(1 + \pi(1 + 4\pi)) = 137.03630037758 \tag{1.11}$$

From Aether Theory in 1972[23]

$$\alpha^{-1} = 108\pi(8/1843)^{1/6} = 137.03591484 \tag{1.12}$$

From Pythagorean theorem in 1975[24]

$$\alpha^{-1} = (137^2 + \pi^2)^{1/2} = 137.0360157199 \tag{1.13}$$

From $N = 42$ invariance groups in 1976[25]

$$\alpha^{-1} = N(N-1)/4\pi = 137.032406 \tag{1.14}$$

In 2004, Stoyan proposed $\alpha = \frac{2}{(n^2+2\pi^2)^{1/2}+n}$, and $\alpha^{-1} = \frac{137+\sqrt{137^2+2\pi^2}}{2} = 137.03601098839$ when $n = 137$ for the regular Tetrahedron geometry.[26] It is one solution from $x^2 - 137x - \pi^2/2 = 0$ but the other one is ignored. They are all in line with the concept that '*all universal constants involve only integral numbers and* π'. The 4π in (1.11) was interesting but went nowhere.[1]

In 2000, Kosinov suggested the more complex but more accurate formula yielding $\alpha^{-1} = 137.03600982$.[27]

$$\alpha^{20} = (\pi\Phi^{14})^{1/13}10^{-43} \tag{1.15}$$

In 2005, Heyrovska proposed[28]

$$\alpha^{-1} = 360\phi^2 - 2\phi^3 = 137.0356280 \tag{1.16}$$

In 2006, Naschia suggested a formula in the E-Infinity Cantorian space-time.[29]

$$\alpha^{-1} = 20\Phi^4 = 137.0820393 \tag{1.17}$$

This type of formula followed Bohr's thought, involving the golden mean, Fibonacci and Lucas numbers.

1-4 Recent Fittings by Gilson and Others

In 1990s, Gilson developed a well-known trigonometric functions, based on the wave capture circling analysis. In order to fit CODATA-2006 $\alpha^{-1} = 137.035999679(94)$,

$$\alpha = \frac{\cos(\pi/137)}{137}\frac{\tan(\pi/(29\cdot 137))}{\pi/(29\cdot 137)} = 1/137.0359997867 \tag{1.18}$$

---

[1]First presented by E. D. Reilly, Jr. in 1971. Note $\beta = \prod_{k=0}^{2}(4\pi - k/\pi) = 4\pi(4\pi - \frac{1}{\pi})(4\pi - \frac{2}{\pi}) = 1836.1517$ and $\beta_{n/e} \approx \beta_{p/e} + \ln(4\pi) = 1838.6827635$ link to $\mu_0 = 4\pi \times 10^{-7}$[Wb/Am] and $k_e = 1/4\pi\varepsilon_0$[Nm^{2}C^{-2}]

where involve two primes 29 and 137 in 2007.[30] However, CODATA-2010 gives $\alpha = 1/137.035999074(44)$. Kirakosyan gives a similar analysis $\sum I_m/I = \sin\triangle\varphi = \alpha^{1/2}$ and $\alpha^{-1} = \frac{1}{\sin^2\triangle\varphi}$ in 2011.[31] [2]

In 2007, Lestone used a blackbody radiation model to get[32]

$$\alpha^{-1} = 16\pi^3/3\zeta(3)g(\sigma_T) = 137.042999 \tag{1.19}$$

In 2009, Markovitch suggested[33]

$$\begin{aligned}\alpha^{-1} &= \frac{A^a - D^d}{B^b} + C^c - E^e = 137.036 \\ &= \frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = \frac{22^3 - 2^{-1}}{25^2} + 11^2 - 1^{-1}\end{aligned} \tag{1.20}$$

In 2010, Rhodes suggested $\alpha^{-1}_{Rhodes} = 4\alpha_{swe} = 4\frac{m_{\nu_\mu}}{m_{\nu_e}}$ with Precision Cryptographic Calculation using the *prime* sequence to obtain the magic number[34]

$$\begin{aligned}\alpha^{-1} &= 4 \cdot \frac{2\cdot 3^2\cdot 5^2\cdot 7\cdot 11\cdot 17\cdot 19\cdot 31\cdot 47\cdot 53\cdot 59\cdot 61\cdot 73\cdot 79\cdot 103\cdot 109\cdot 113\cdot 131\cdot 149}{2\cdot 13\cdot 23\cdot 29\cdot 37\cdot 41\cdot 43\cdot 67\cdot 71\cdot 83\cdot 89\cdot 97\cdot 101\cdot 107\cdot 127\cdot 137\cdot 139\cdot 151} \\ &= 4\times 4.44\times 10^{29}/1.52\times 10^{31} = 137.0359991047437444154\end{aligned} \tag{1.21}$$

However, according to the experimental $\frac{m_{\nu_\mu}}{m_{\nu_e}} = \frac{0.8019meV}{27.45meV} = 34.2312$, then $\alpha^{-1}_{Rhodes} = 136.9248$.

In 2011, Code gives[35]

$$\begin{aligned}\mathcal{L}(\Phi,\psi) &= \Phi^{\log(\psi^{\log(\Phi^{\log(\psi^{\log(\Phi^{\log(\psi)})})})})} \\ \psi &= \triangle - 137 + 137(\Phi+\phi)Im\left(\frac{i^{i^{\cdots}}}{1+\frac{1}{2+\frac{1}{\ddots}}}\right)\end{aligned} \tag{1.22}$$

[2] Other approximate trigonometric functions include: $\tan^{-1}(\alpha^{-1}) \simeq \frac{\pi}{2}$; $\cos^{-1}(\alpha) \simeq \frac{\pi}{2}$; $\cos(\alpha^{-1}) \simeq e^{-1}$; $\sin(\alpha) \simeq \alpha$.

where $\Phi = 1.6180$, and $\phi = 0.6180$, and $\triangle = A^{-1} - 8A^{-(\triangle^{-1}(\triangle)=33+\frac{4}{137}+\frac{144000}{1209485})}$ and $A = e^{\frac{1}{12}-\zeta'(-1)} = 1.28242712\cdots$, then $\alpha = \mathcal{L}(\phi, \psi) = 1/137.03599908573\cdots$ matches $\alpha_{QED}^{2008} = 1/137.035999084(51)$.

In 2012, Schonfeld proposed a simple but less accurate formula $\alpha^{-1} = \pi^4\sqrt{2}m_{qm}/m_e$ with the bare charge equal to[36]

$$\alpha^{-1} = \sqrt{2}\cdot\pi^4 = 137.757 \tag{1.23}$$

Poojary published $\alpha^{-1} = \sqrt[3]{\frac{7}{18013664}}$, which matches exactly to value $(7.297352533(27) \times 10^{-3})$ recommended by CODATA 1998.[36]

All the precision study on 137 started after Eddington's *adding-one* formula based on matrix theory in the 1930s. He believed $(10\times 10)+(6\times 6) = 136$ real matrix and $(10 \times 6) + (6 \times 10) = 120$ imaginary matrix existed in spacetime, and also proposed an additional matrix to match with the experimental value about 137, then obtained a whole number

$$\begin{aligned} \alpha^{-1} &= \frac{n^2(n^2+1)}{2} + 1 \\ &= \mathbf{136 + 1 = 137} \end{aligned} \tag{1.24}$$

where $n = 4$. He later proposed $\alpha^{-1} = 137\beta^{1/24} = 137.042$ in 1948.[37]

We all learned valuable lessons from the early explorers. After countless efforts, Pauli's simplest question still remains unanswered: "Why 137?"[38, 39, 40, 41, 42] In his Nobel Lecture delivered in Stockholm on 13 December 1946, Pauli expressed his goal was to establish a theory "*which will determine the value of the fine-structure constant and will thus explain the* ***atomistic structure of electricity****, which is such an essential quality of all atomic sources of electric fields actually occurring in nature.*"[43] As the initialization, "*... **from a physical point of view, that the existence of atomicity, in itself so simple and basic, should also be interpreted in a simple and elementary manner by theory and should not, so to speak, appear as a trick in analysis.***"[43] His lifelong search for 137, a millennium puzzle, ended in hospital room 137.[44]

1-5 The Various Experimental Data

The difficulty of finding the correct α formula is partly due to the uncertainty of the experimental values - approximately 137.036. Some experimental data of the fine structure constant is listed in the **Table 1.1**.[45, 46, 47, 48,49, 50, 95]

Table 1.1 The history of the experimental Fine Structure Constant Values

DATE	$1/\alpha$	SOURCE
1916	137.360563948	A. Sommerfeld
1929	137.29 ± 0.11	R. Birge
1930	136.94 ± 0.15	W. Bond
1932	137.305 ± 0.005	R. Birge
1935	137.04 ± 0.02	F. Spedding et al.
1941	137.030 ± 0.016	R. Birge
1943	137.033 ± 0.092	U. Stille
1949	137.027 ± 0.007	J. DuMond, E. Cohen
1949	137.041 ± 0.005	H. Bethe, C. Longmire
1957	137.0371 ± 0.0005	J. Bearden, J. Thomsen
1969	137.03602(21)	CODATA 1969
1973	137.03612(15)	CODATA 1973
1987	137.0359895(61)	CODATA 1986
1998	137.03599883(51)	T. Kinoshita
2000	137.03599976(50)	CODATA 1998
2002	137.03599911(46)	CODATA 2002
2007	137.035999070(98)	G. Gabrielse et al.
2008	137.035999679(94)	CODATA 2006
2008	137.035999084(51)	D. Hanneke et al.
2010	137.035999037(91)	R. Bouchendira
2010	137.035999074(44)	CODATA 2010
2012	137.035999174(35)	T. Aoyama et al.

Many people are trying to fit the latest data mathematically, but fail to understand that CODATA is a statistic of experimental data and keeps

changing. The QED calculation itself is a supercomputer numeral fitting and also keeps changing. The reality of experimental data is that the fine structure constant measured using different methods will create slight differences around 137.036.[46, 48, 49,50]

$$\begin{aligned}
\alpha^{-1}(Muonium\,Hyperfine) &= 137.0359997(84) \quad [61ppb] \\
\alpha^{-1}(ac\,Josephson) &= 137.0359875(43) \quad [31ppb] \\
\alpha^{-1}(Quantum\,Hall) &= 137.0360030(25) \quad [18ppb] \\
\alpha^{-1}(Neutron\,Wavelength) &= 137.0360077(28) \quad [21ppb] \\
\alpha^{-1}(Atom\,Interferometry) &= 137.0360001(11) \quad [7.7ppb] \\
\alpha^{-1}(Optical\,Lattice) &= 137.03599883(91) \quad [6.7ppb] \\
\alpha^{-1}(Electron\,g/2\,QED) &= 137.035999173(35) \quad [0.35ppb]
\end{aligned} \tag{1.25}$$

After 2006, however, CODATA started using α to calibrate all other physical constants. Therefore, any experiment using CODATA values will retrieve the same CODATA recommended α value. We should not be fooled by those new reports of testing QED.[50] It is the same circular logic of discovering eggs during an Easter egg hunt.

Whether α is variable with time or location in the universe is still in debate. Even stranger, the high-energy level $\alpha(m_z^2)$ is approximately 1/128 compared with the fermion zero-energy value of approximately 1/137.[47]

The α value obtained by the electron g-2 QED in (1.25) is an experiment dependent value, not a theoretical value. Therefore, QED can not offer a true explanation for 137. The *true* α must have a mathematical underpinning with an applicable physical definition, otherwise even the best experiments still can not answer "Why 137?"

Born pointed out that "*If α is a mathematical number, the constant $\mathbf{e}, \hbar, c$ are not independent! In other words, from measurement of two of them the third can be calculated.*" "*If α were bigger than it really is, we should not be able to distinguish matter from ether, and our task to disentangle the natural laws would be hopelessly difficult. The fact however that alpha has just its value 1/137 is certainly no chance but itself a law of nature. It is clear that the explanation of this number must be the central problem of natural philosophy.*"

1-6 The Clues from Einstein and Feynman

Einstein was a pioneer on searching for an α math formula started in 1909. He said in 1945: "*There are two kinds of constants: apparent and real ones. ... I therefore believe that such number can be only of a basic type, as for instance* π *or* e *...*"[51]

In 1954, Turing wrote "Charge= $\frac{e}{\pi}$ arg of character of a 2π rotation" without formulation.[53]

In 1985, Feynman said in lecture "QED: The Strange Theory of Light and Matter"[52]

"*If electrons were ideal, ... J would be simply be its* ***"charge"*** *(the amplitude for the electron to couple with a photon). ...*

... J - ***theoretical numbers that are not directly observable anyway****; ...*

But no such ideal electrons exist. ... a real electron, which emits and absorbs its own photons from time to time, and therefore depends on the amplitude for coupling, J. ...

... the observed coupling constant **e** *-* ***the amplitude for a real electron to emit or absorb a real photon.***[3] *It is* ***a simple number*** *that has been experimentally determined to be close to* -0.08542455. (*My physicist friends ... remember it as the inverse of its square: about* 137.03597 ...).

... a ***magic number*** *that comes to us with no understanding by man ...*

A good theory world say the **e** *is the square root of 3 over 2 pi squared, or something. There have been, from time to time, suggestions as to what* **e** *is,* ***but none of them has been useful****. ... Every once in a while, someone notices that a certain combination of* ***pi****'s and* **e***'s (the base of the natural logarithms), and* ***2****'s and* ***5****'s produces the mysterious coupling constant, but it is a fact not fully appreciated by people who play with arithmetic that you would be surprised how* ***many*** *numbers you can make out of pi's and e's and so on. ...*

Even though we have to resort to ***a dippy process*** *to calculate J today,*

---

[3] where $e = \alpha^{1/2}$ is related to J - "*theoretical numbers that are not directly observable anyway*" - assumed by Feynman [52] p128.

it's possible that someday a legitimate mathematical connection between J and e will be found. That would mean that J is ***the mysterious number, and from it comes e*** "[52]

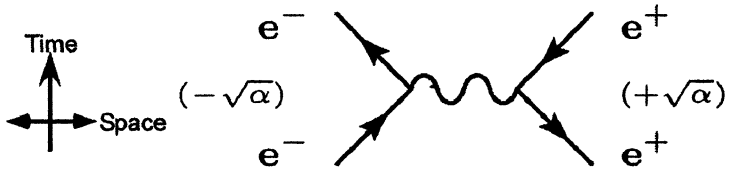

Fig. 1.1: The electron e^- and positron e^+ as time reversal in the Feynman diagram are given as the coupling constants $\pm\alpha^{1/2} = \pm 0.08542455$

Like Pauli and Dirac, Feynman was never satisfied with any math formula for the fine structure constant, but he certainly investigated some different approaches:

(1) Studying $-\alpha^{1/2} = -0.08542455$ or $-\alpha^{-1/2} = -11.70623667\ldots$ instead of $\alpha = 0.007297\ldots$ or $\alpha^{-1} = 137.036\ldots$. (i.e., the ratio $\pm e/q_{Plack}$ instead of $e^2/\hbar c = (e/q_{Plack})^2$).

(2) $\alpha^{1/2}$ is a simple number as a Lorentz invariant scalar quantity, but not a vector or a complex number.

(3) $\alpha^{1/2}$ is the physical coupling constant: a real number for a real electron to emit or absorb a real photon.

(4) There is a $\pm$ sign on $\alpha^{1/2}$ for the *positive* and *negative* charge distinguished by Franklin in 1750s, which is clearly presented as the time reversal in the Feynman diagram (**Fig. 1.1**).

(5) The "magic number" is for the creation of charge and works forever, so that it could be a math limitation from the infinite series/products, or continued fraction, etc.

(6) $\alpha^{1/2}$ may be obtained using a certain combination of π, e, 2 and 5 (e.g., the golden ratio $\Phi = \frac{1}{\phi} = 2\cos(\frac{\pi}{5})$).

(7) There are many π *and* e combinations which are not unique solutions

to this important unique physical number.[4]

(8) The α math formula must be useful for solving physical mysteries, mere arithmetic is not appreciated.

(9) There may be an unique mysterious theoretical number J for the unique physically measurable $\pm\alpha^{1/2}$.

(10) No ideal electrons exist. A real electron depends on the coupling, J.

(11) QED calculation is "a dippy process", and we need to probe a legitimate mathematical connection between J and $\pm\alpha^{1/2}$.

Soon after Feynman's lecture, we noticed a simple formula,

$$\alpha^{1/2} \approx \pm\frac{\log e}{\Phi\pi} = \log e^{\pm\phi/\pi} = \frac{\pm 1}{\Phi\pi \ln 10} = 1/\ln 10^{\pm\Phi\pi} = \pm 0.0854372111 \tag{1.26}$$

which yields $\alpha = [\frac{\phi}{\pi}\log e^{\pm 1}]^2 = \frac{1}{136.995532}$, out of range of the experimental values. However, it is embedded with the $\pm$ sign and only involves three basic math constants $\Phi - \phi - e - \pi$ in Euler-type Identity $e^{\pm i\pi} + \Phi = \phi$ ("the most beautiful equation" - Feynman (**Fig. 1.2**)). In this paper, we put this type of formula on debate.

2 Probing of the Fine Structure Constant

2-1 Maxwell Equations and Charge-Pole Quantization

Dimensionless physical constants such as the fine-structure constant $\alpha = \frac{\mathbf{e}^2}{2\varepsilon_0 hc} = \frac{1}{137.036}$ are not simply mathematical coincidences; their values are governed by the deepest numerical theory.[18] The *true* α must have a mathematical underpinning with an applicable physical definition, otherwise even the best experiments still can not answer Pauli's question: "Why *137*?"[43,

[4] Beyond $\sqrt{3}/2\pi^2 = 0.0877467$, we can also get $e/\pi^3 = 0.0876687$, $\tan^{-1}(e,\pi)/\pi^2 = 0.08688410$, $e + e^{-1} - 3 = 0.0861612$, $2/e^2\pi = 0.0861571$, $\arg(e + i\pi)/2\Phi\pi = 0.084329$, and $e\pi\% = \frac{e\pi}{100} = \frac{e}{10}\frac{\pi}{10} = \frac{e}{2\times 5}\frac{\pi}{2\times 5} = \frac{e}{2}\frac{\pi}{2}\frac{1}{(\Phi+\phi)^4} = 0.08539734222$, *etc.* The last one is close to $\alpha^{1/2}$ and $\alpha^{-1}_{e\pi\%} = 137.1233109$ with $\mathbf{e} = \sqrt{\alpha\hbar c} = \frac{e\pi}{100}\sqrt{\hbar c} = \frac{e}{100}\sqrt{\frac{\pi}{2}hc}$ in [esu].

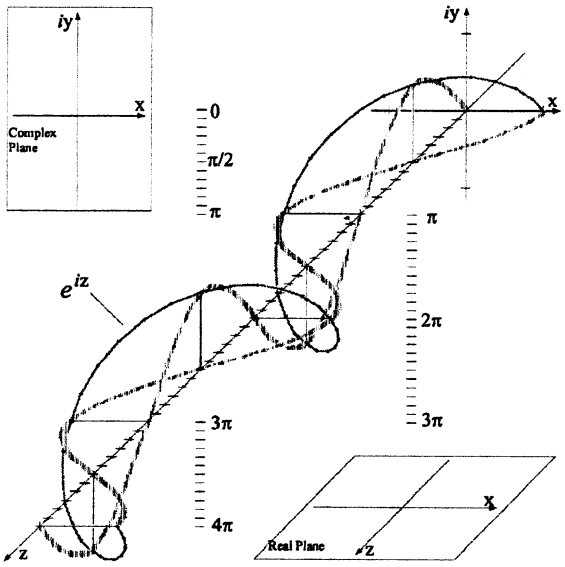

Fig. 1.2: Three-dimensional visualization of Euler's formula - The circular polarization

54] Let's start from the free-space characteristics of the electromagnetic wave.

In 1856, Weber and Kohlrausch experimentally discovered that $1/\sqrt{\varepsilon_0\mu_0}$ was very close to the light-speed c measured by Fizeau in 1849 and confirmed by Foucault in 1962.[55] This is finally theoretically approved by the Maxwell equations in 1864.[56]

In the geometric algebra $\boldsymbol{Cl_n}$ with an orthogonal basis $\mathcal{G}(p,q)$ consists of $\sum_{k=0}^{n}\binom{n}{k} = 2^n = 2^{p+q}$ elements, such as $\mathcal{G}(3,0)$ for the 3D Euclidean space with $2^3 = \{1 \oplus 3 \oplus 3 \oplus 1\}$,[57]

$$\{\underbrace{1}_{scalar}, \underbrace{e_1, e_2, e_3}_{vectors}, \underbrace{e_1e_2, e_1e_3, e_2e_3}_{bivectors}, \underbrace{e_1e_2e_3}_{trivector}\} \tag{2.1}$$

the Maxwell four-equations are united into a single equation in the *SI* system

$$\underset{Gauss(E)}{(\nabla\cdot\mathbf{E}-\tfrac{\rho}{\varepsilon_0})}+Ic\underset{Gauss(M)}{(\nabla\cdot\mathbf{B})}+I\underset{Faraday}{(\nabla\times\mathbf{E}+\tfrac{\partial\mathbf{B}}{\partial t})}-c\underset{Ampère-Maxwell}{(\nabla\times\mathbf{B}-\tfrac{1}{c^2}\tfrac{\partial\mathbf{E}}{\partial t}-\mu_0\mathbf{J})}=0 \tag{2.2}$$

i.e., $0+0+0-0=0$ in the light-cone of 4D Minkowski space $\{x,y,z,ict\}$. The electric field and magnetic field are co-existed at the certain point of spacetime. From (2.2),

$$[\frac{1}{c}\partial_t-\nabla](\mathbf{E}+I\mu_0c\mathbf{H})-(\frac{\rho}{\varepsilon_0}+\mu_0c\mathbf{J}) \;=\; 0 \tag{2.3}$$

where $\nabla\mathbf{X}=\nabla\cdot\mathbf{X}+I\nabla\times\mathbf{X}$ for spatial vectors $\mathbf{X}$, the pseudo-scalar $I^2=-1$ commutes with all spatial vectors; the electric constant $\varepsilon_0\approx 8.854187817\ldots\times10^{-12}[\frac{\mathrm{F}}{\mathrm{m}}]=\frac{10^7}{4\pi c^2}[\frac{\mathrm{F}}{\mathrm{m}}]\approx\frac{10^{-9}}{36\pi}[\frac{\mathrm{s}}{\mathrm{m}\Omega}]$ and the magnetic constant $\mu_0=1.2566370614\ldots\times10^{-6}[\frac{\mathrm{H}}{\mathrm{m}}]=4\pi\times10^{-7}[\frac{\mathrm{s}\Omega}{\mathrm{m}}]$, $\varepsilon_0\mu_0c^2=\sigma_0Z_0=1$ with $c=299792458[\frac{\mathrm{m}}{\mathrm{s}}]$. Since Ω and $\frac{1}{\Omega}$ are reciprocals in $[\frac{\mathrm{H}}{\mathrm{m}}]=[\frac{\mathrm{s}\Omega}{\mathrm{m}}]$ and $[\frac{\mathrm{F}}{\mathrm{m}}]=[\frac{\mathrm{s}}{\mathrm{m}\Omega}]$, the characteristic impedance or conductance of vacuum is the ratio of electric and magnetic field strength $(\mathbf{E},\mathbf{H})$ or flux density $(\mathbf{D},\mathbf{B})$

$$\begin{aligned} Z_0 &= \mu_0c=\sqrt{\frac{\mu_0}{\varepsilon_0}}=\frac{|\mathbf{E}|}{|\mathbf{H}|}=\frac{|\mathbf{B}|}{|\mathbf{D}|}=376.730313461\ldots\ [\Omega] \\ \sigma_0 &= \varepsilon_0c=\sqrt{\frac{\varepsilon_0}{\mu_0}}=\frac{|\mathbf{H}|}{|\mathbf{E}|}=\frac{|\mathbf{D}|}{|\mathbf{B}|}=2.6544187\times10^{-3}\ [\mathrm{S}] \end{aligned} \tag{2.4}$$

where $\mathbf{E}=\mathbf{D}/\varepsilon_0[\frac{\mathrm{V}}{\mathrm{m}}]$ and $\mathbf{H}=\mathbf{B}/\mu_0[\frac{\mathrm{A}}{\mathrm{m}}]$ are the amplitude of the electric and the magnetic field; $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux densities. The electromagnetic energy density is $u=\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})=\frac{1}{2}(\varepsilon_0\mathbf{E}^2+\mu_0\mathbf{H}^2)$ and $\mathbf{E}=c\mathbf{B}$ as the Faraday law. The intensity of radiation $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ is called Poynting vector with the right-hand rule. The asymmetric electromagnetic field $\mathbf{E}>\mathbf{H}$ $(\mathbf{D}<\mathbf{B})$ in **Fig. 2.1** is elliptical but near linear (the ratio of semi-axis is $376.7:1\approx5!\pi:1$)[5]

[5]The asymmetric vacuum properties are eliminated in Heaviside-Lorentz units $c=\varepsilon_0=\mu_0=\sigma_0=Z_0=1$. If $\hbar=c=1$ then charge $e=\sqrt{4\pi\alpha}=0.302822$, which is used in particle physics with a mystery of symmetric broken. The Lorentz-Maxwell equations use the micro-fields $\mathfrak{e}$ and $\mathfrak{h}$ for the single charged particle.

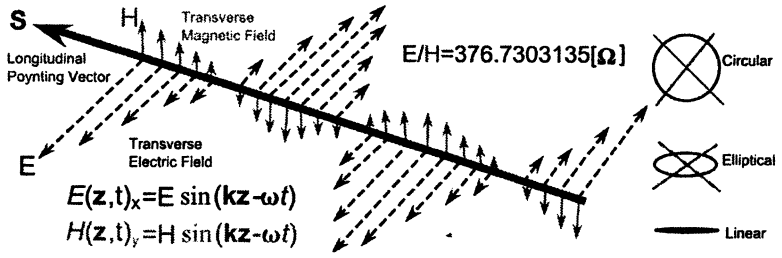

Fig. 2.1: Electromagnetic field with $\mathbf{E} : \mathbf{H} = \mathbf{B} : \mathbf{D} = 376.730 : 1$, it is a near linear polarization.

In the electromagnetic field, an electric elementary charge is accelerated by a Lorentz force $\mathbf{F}_l = \mathbf{e}[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$ without the radiation of photon $h\nu$. The elementary charge $\mathbf{e}$ is defined by the Planck constant h and the fine structure constant α with the electromagnetic properties of free space in (2.4).[58] The α serves as a bridge in the quantum h to electricity $\mathbf{e}$, relativity c, and vacuum $\varepsilon_0, \mu_0, \sigma_0, Z_0$.[59] $\frac{\mathbf{e}^2}{h}$ are particle quanta properties and $\sigma_0 = \varepsilon_0 c$ are spacetime properties in the fine structure constant. Experimentally,

$$\begin{array}{llll} \mathbf{e} & = [2\alpha h\varepsilon_0 c]^{1/2} & = [2\alpha h/\mu_0 c]^{1/2} & = [2\alpha\sigma_0 h]^{1/2} & = [2\alpha h/Z_0]^{1/2} \\ & = G_0/K_J & = 2/K_J R_K & = G_0\Phi_0 & = 2\Phi_0/R_K \end{array} \tag{2.5}$$

where $K_J = 2\mathbf{e}/h = 1/\Phi_0 = 483597.870(11) \times 10^9\,[\mathrm{HzV}^{-1}]$ is the Josephson constant for the superconductor quantization and $R_K = h/\mathbf{e}^2 = 2/G_0 \overset{\text{def}}{=} 25812.807\,[\Omega]$ is the von Klitzing constant for the quantum Hall quantization, $G_0 = 2\mathbf{e}^2/h = 7.7480917346\,(25) \times 10^{-5}[\mathrm{S}]$ is the conductance quanta and $\Phi_0 = h/2\mathbf{e} = 2.067833758\,(46) \times 10^{-15}\,[\mathrm{Wb}]$ is the magnetic flux quanta. From (2.5), the quantum impedance and conductivity are linked to (2.4) by the fine structure constant

$$\begin{aligned} R_K &= \frac{h}{\mathbf{e}^2} = Z_0/2\alpha = \sigma_K^{-1} = 25812.807\ [\Omega] \\ \sigma_K &= \frac{\mathbf{e}^2}{h} = 2\alpha \cdot \sigma_0 = R_K^{-1} = 3.8740 \times 10^{-5}[\mathrm{S}] \end{aligned} \tag{2.6}$$

Charged particles with high quantum impedance R_K (or low conductivity σ_K) are surrounded by a low impedance of free-space Z_0 (or high conductivity σ_0), and the coupling constant between two zones (Fermion/Boson) is the fine structure constant. In this way, a charged particle becomes a space-time light-trap for the electromagnetic wave. This is the foundation of the photoelectric effect.

The reality of Ohm's law on the atomic scale is a memristor[60]

$$M(q) = \frac{V(t)}{I(t)} = \frac{d\Phi_m}{dQ} = \frac{d\Phi_B}{dq} = \frac{-d\Phi_B/dt}{-dq/dt} \tag{2.7}$$

A memristor is a functional relationship between the time integral of current (related to charge) and the time integral of voltage (related to magnetic flux). During the study of 137, Dirac proposed charge-monopole quantization $\mathbf{e}g = \frac{1}{2}nhc$ in *cgs* unit, which can be expressed as $\mathbf{e}g = nh$ in *SI* unit with $n \in \mathbb{Z}$.[6] **Resistance is the ratio of the electric and magnetic properties**, i.e., $R = M(q)$ as the constant ratio of the charge $\mathbf{e}$[C] and the magnetic monopole $g = 2\Phi_0$[Wb].

$$R = M(q) = \frac{d\Phi_B}{dq} = \frac{-\mathbf{B}d\mathbf{S}/dt}{-2\mathbf{e}dN/dt} = \frac{1}{2N}\frac{h}{\mathbf{e}^2} = \frac{Z_0}{4\alpha N} = \frac{R_K}{2N} = \frac{1}{2N}\frac{g}{\mathbf{e}} \tag{2.8}$$

where the magnetic flux $d\Phi_B = \mathbf{B}d\mathbf{S}$ and $R_K = \frac{g}{\mathbf{e}} = \frac{h}{\mathbf{e}^2}$. The magnetic coupling constant is defined by the monopole as $\beta_m = \frac{g^2}{4\pi\mu_0\hbar c} = \frac{\hbar c}{4\mathbf{e}^2} = \frac{1}{4\alpha} = \frac{\sigma_0}{G_0} = 34.259$ (the *magnetic "fine structure" constant*), a magnitude for the unified strong-electroweak coupling constant is defined as $\alpha_{swe} = \frac{m_{\nu_e}}{m_{\nu_\mu}} \approx 34.26^{-1}$.[61]

2-2 Alpha Formula, Charge-sign and Criterion

Feynman talked about $\alpha^{1/2} \sim -0.08542455$ as the number "my physicist friends won't recognize...",[52] because it is better known as $\alpha^{-1} \sim$

[6] J. D. Jackson, Classical electrodynamics. 3rd edition, (1998) p275 $\frac{\mathbf{e}g}{4\pi\hbar} = \frac{\alpha g}{Z_0\mathbf{e}} = \frac{n}{2}$ ($n = 0, \pm1, \pm2, \ldots$) (6.153)

137.036.[11] In this paper, we show how I/FQHE exposes the physical information formula

$$|\alpha^{1/2}| \equiv \pm\frac{M\phi}{K\pi} \equiv \frac{\log e^{\pm 1}}{K\Phi\pi} \equiv \log e^{\pm\phi/K\pi} \equiv \frac{1}{\ln 10^{\pm K\Phi\pi}} \tag{2.9}$$

where $M = \log e = 1/\ln 10$ is used in the logarithmic *information entropy* $S = \log e^{w}$ or $S = \ln 10^{w}$ in *units* [ban], [hart], [nat], $\Phi - \phi - e - \pi$ each only appear once and can be presented as continued fractions

$$\begin{aligned} \Phi &= 1 + \phi = 1 + \tfrac{1}{1+}\tfrac{1}{1+}\cdots = 1.61803\ldots \\ e &= 2 + \tfrac{1}{1+}\tfrac{1}{2+}\tfrac{2}{3+}\tfrac{3}{4+}\tfrac{4}{5+}\cdots = 2.71828\ldots \\ \pi &= 3 + \tfrac{1^2}{6+}\tfrac{3^2}{6+}\tfrac{5^2}{6+}\tfrac{7^2}{6+}\tfrac{9^2}{6+}\cdots = 3.14159\ldots \end{aligned} \tag{2.10}$$

in the Euler-type Identity

$$e^{\pm i\pi} + \Phi = \phi \tag{2.11}$$

From (2.9), one gets

$$\alpha \equiv \frac{\mathbf{e}^2}{2\varepsilon_0 hc} \equiv \frac{\mu_0 c\mathbf{e}^2}{2h} \equiv \left(\frac{\log e^{\pm 1}}{K\Phi\pi}\right)^2 \equiv \log^2\left(e^{\pm 1/K\Phi\pi}\right) = \frac{\log e^{\pm\phi^2}}{\ln 10^{\pm K^2\pi^2}} \tag{2.12}$$

There are only two types of elemental charge, $\mathbf{e}^-$ and $\mathbf{e}^+$, or their combination 0, distinguished by Franklin.[62] Feynman found the time reversal, but the *intrinsic* difference between $\mathbf{e}^-$ and $\mathbf{e}^+$ is still a mystery. The elementary charge is related to $\alpha^{1/2} = \log e^{\pm 1/K\Phi\pi}$ in (2.9). We know Euler's number (the infinite series was first published by Newton in 1669)

$$\begin{aligned} e^{+1} &= \Gamma(1,-1) = \sum_{n=0}^{\infty}\frac{(+1)^n}{n!} = 2.71828\cdots \\ e^{-1} &= \Gamma(1,+1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = 0.36787\cdots \end{aligned} \tag{2.13}$$

The $\pm$ sign is auto-formed by $\log e^{\pm 1} = \pm 0.43429$, where the stable negatively charged electron involves an *alternating* series $\frac{1}{e} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = \sum_{2n}^{\infty}\frac{1}{n!} - \sum_{2n+1}^{\infty}\frac{1}{n!} = 1.543080635 - 1.175201194 = 0.367879441$. The photon is an energy quanta with light-speed c and the electron is a slower host (αc) during

the photon-electron interaction. It is well known that Euler's number e is linked to the *time* related growth or decay (i.e., time reversal), and to Fibonacci numbers as (alternating $\pm$ at the *odd* factorial term)[7]

$$e^{\pm 1} = \frac{\sum_{k=0}^{\infty} \frac{F(1\pm k)}{k!}}{\sum_{k=0}^{\infty} \frac{F(1\mp k)}{k!}} = \frac{\frac{1}{0!} \pm \frac{1}{1!} + \frac{2}{2!} \pm \frac{3}{3!} + \frac{5}{4!} \pm \frac{8}{5!} + \frac{13}{6!} \cdots}{\frac{1}{0!} \mp \frac{0}{1!} + \frac{1}{2!} \mp \frac{1}{3!} + \frac{2}{4!} \mp \frac{3}{5!} + \frac{5}{6!} \cdots} \tag{2.14}$$

In this way, the elementary charge is frozen in time in the quantum theory and embedded with the $\pm$ sign using logarithms (i.e., $\log \lim_{n\to\infty} (\frac{n}{\sqrt[n]{n!}})^{\pm 1} = \log \lim_{n\to\infty} (1+\frac{1}{n})^{\pm n} = \pm \log e$). More importantly, $e^{\pm 1}$ is statistically an algebraic sum of the arrangement ratio (permutation) between Bosons (spin-1 for photon) and Fermions (spin-$\frac{1}{2}$ for electron), which compel a real electron to emit and absorb a real photon.[52, 54]

$$\begin{aligned} e^{\pm 1} &= \frac{(\pm 1)^0}{0!} + \frac{(\pm 1)^1}{1!} + \frac{(\pm 1)^2}{2!} + \frac{(\pm 1)^3}{3!} + \cdots + \frac{(\pm 1)^n}{n!} \cdots \\ &= \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{n!} = \sum_{n=0}^{\infty} (\pm 1)^n \frac{\text{Arrangement of Bosons}}{\text{Arrangement of Fermions}} \end{aligned} \tag{2.15}$$

Bosons are indistinguishable particles, with only ***one*** arrangement no matter how many photons are in the box, while the n distinguishable Fermions (Pauli Principle) must have $n!$ arrangement.[64] In (2.15), the negatively charged electron has an *alternating* series, while the positively charged particle takes a *normal* series.

The arbitrary charge sign + or $-$, given by Franklin in 1750, is interpreted by using the different intrinsic statistics in (2.13)-(2.15). It self-dictates the $\pm$ sign for the charge conservation, e.g., the annihilation of electron and positron ($\mathbf{e}^- + \mathbf{e}^+ = 2\gamma$) or the formation of a Cooper pair ($\mathbf{e}^- + \mathbf{e}^- = 2\mathbf{e}^-$).

$$\begin{aligned} \mathbf{e}^{\pm} + \mathbf{e}^{\mp} &= (2\sigma_0 h)^{1/2} \log e^{\pm\phi/K\pi \mp \phi/K\pi} = 0 \\ \mathbf{e}^{\pm} + \mathbf{e}^{\pm} &= (2\sigma_0 h)^{1/2} \log e^{\pm 2/K\Phi\pi} = 2 \cdot \mathbf{e}^{\pm} \end{aligned} \tag{2.16}$$

[7] Note $\phi = \lim_{n\to\infty} \frac{F_n}{F_{n+1}}$, $\Phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 2\cos(\frac{\pi}{5})$, and $\Phi^n = \Phi^{n-1} + \Phi^{n-2} = \Phi F_n + F_{n-1}$.

(2.16) sets up a simple criterion for the $\alpha^{1/2}$ math formula. The elementary charge is a quanta for an electron particle in the point charge model, and the math formula of $\alpha^{1/2}$ must allow for an algebraic sum. ***Charge sign*** with ***intrinsic characteristics*** and ***conservation*** are three simple criterion for approving the $\alpha^{1/2}$ math formula. This criterion can be used to exclude those mathematical coincidences of the *137* formulas.

2-3 I/FQHE and Variable Charge-Pole Ratio

The elementary charge changes the vacuum property of an electromagnetic field. In fact, $4\alpha = 0.02918941 = 1/34.259$ for the strong interaction is equal to the conductivity ratio

$$4\alpha = Z_0 G_0 = \frac{G_0}{\sigma_0} \equiv \frac{2\cdot \mathbf{e}^2}{\sigma_0 h} \equiv \frac{2Z_0\mathbf{e}^2}{h} \equiv \left(\frac{2}{\pi}\frac{\log e^{\pm 1}}{K\Phi}\right)^2 \tag{2.17}$$

where the conductance quantum $G_0 = \frac{2\mathbf{e}^2}{h}$ is related to the von Klitzing constant $R_{\rm K} = \frac{h}{\mathbf{e}^2}$. Klitzing discover $R_H = \frac{h}{n\mathbf{e}^2} = \frac{\mu_0 c}{2n\alpha} = \frac{Z_0}{2n\alpha}$ for IQHE in the *weak* magnetic field.[65] FQHE is still only incompletely described by the theories of 2DEG, Composite Fermion, Topological Order, Anyon, and so on.[66, 67, 68, 69, 70, 71] The acid test for (2.9) is whether it can give a good interpretation of the FQHE of GaAs-GaAlAs in the *strong* magnetic field near 0 K (**Fig. 2.2**). This long and hard procedure is absolutely necessary to establish and confirm the basic structure of the fine structure constant math formula in (2.9).

Quantum Hall conductance $\sigma_{xy} = 1/R_{xy}$ is equal to $\mathbf{e}N/\mathbf{B}$ and the variable reciprocal von Klitzing constant $R_{\rm K}/\nu_i$, one whole charge $\mathbf{e}$ for conducting and another cycling fractional charge $\mathbf{e}_\nu$ for the variable external magnetic field $\mathbf{B}$.

$$\sigma_{xy}^{h/\mathbf{e}} = \frac{1}{R_{xy}} = \frac{\pm\mathbf{e}N}{\mathbf{B}} = \frac{\pm 1}{R_{\rm K}/\nu_i} = \frac{\pm\mathbf{e}}{h/(\nu_i\mathbf{e})} = \frac{\pm\mathbf{e}}{h/\mathbf{e}_\nu} = \frac{\pm\mathbf{e}}{\Phi^\nu} \tag{2.18}$$

where $N \sim 10^{11}\,[\mathrm{cm}^{-2}]$ is the 2D charge density; $\mathbf{e}_\nu = \nu_i\mathbf{e}$ is the fractional charge; $\Phi^\nu = h/\mathbf{e}_\nu$ is a quantized variable linked to the external variable

magnetic field **B** in [Tesla] and to the magnetic flux quanta $\Phi_0 = h/\mathbf{e}$. Coupling of holes (+) and electrons (−) are found on InAs/GaSb, and graphene at room-temperature.[72, 73] The existence of *quasi*-electron and *quasi*-hole states carrying fractional charge $\mathbf{e}_\nu$ are experimentally confirmed.[74] In (2.18), the conducting multi-charges $\mathbf{e}N$ are treated as a single charge **e**, and another cycling charge **e** is further rearranged as an assembling charge $\mathbf{e}_\nu$, in either Integers or Fractions. In (2.18), only the *divisors* (**B**, ν_i, $\mathbf{e}_\nu$, Φ^ν) are variables. $R_{xy} = 1/\sigma_{xy}$ is *proportional* to Φ^ν and *inversely proportional* to ν_i (**Fig. 2.2**).

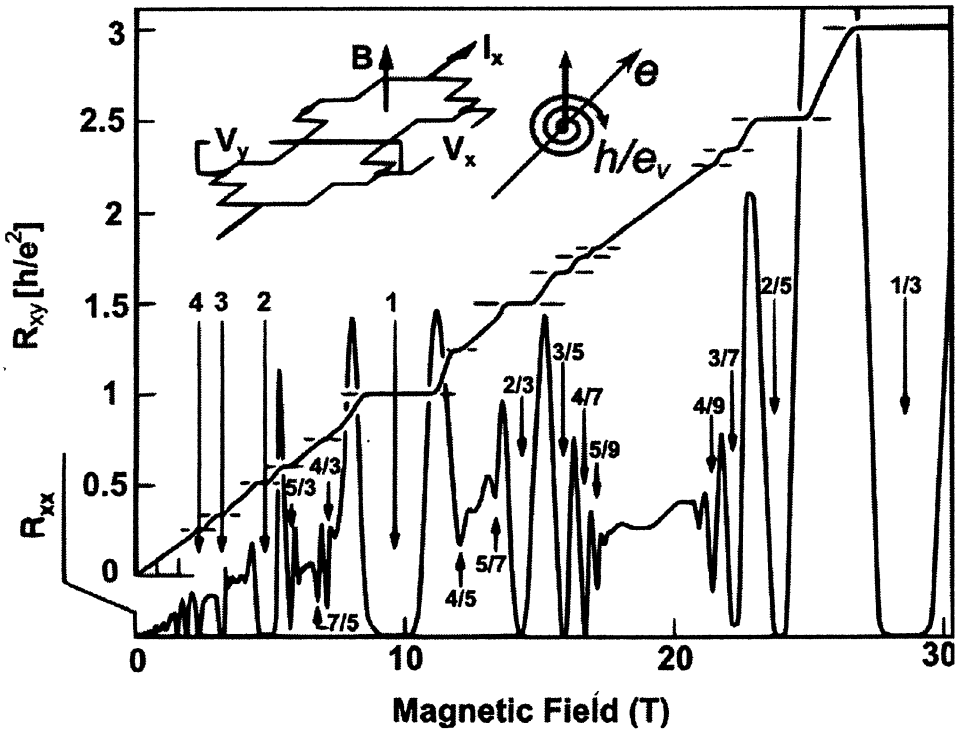

Fig. 2.2: I/FQHE resistance plateaus measured in GaAs-GaAlAs with a perpendicular magnetic field **B** [T]. $R_{xy} = 1/\sigma_{xy}$ is linked to a conducting charge and the cycling fractional charge is restructured by the external magnetic field **B** [T].

Physically, due to Lorentz force, it is easy to assume that if more cycling electrons are trapped in higher external magnetic fields **B**, decreased charge

density N makes R_{xy} increase. Therefore, the cycling fractional charge $\mathbf{e}_\nu$ is not for conducting, but is rather allied to the variable external magnetic field $\mathbf{B}$. From the monopole-charge quantization, the elementary conductance $\sigma = \frac{\mathbf{e}}{g} = \frac{\mathbf{e}^2}{h} = \frac{\mathbf{e}}{2\Phi_0}$ (or the elementary resistance (i.e., the von Klitzing constant) $R_{\rm K} = \frac{g}{\mathbf{e}} = \frac{h}{\mathbf{e}^2} = \frac{2\Phi_0}{\mathbf{e}}$) is the ratio of the elementary charge and the magnetic monopole. From (2.8) and (2.18)

$$\sigma_{xy}^{h/\mathbf{e}} = \frac{1}{R_{xy}} = \frac{\pm\nu_i \mathbf{e}^2}{h} = \frac{2\nu_i\alpha}{Z_0} = 2\nu_i\alpha\sigma_0 = \pm\nu_i\frac{\mathbf{e}}{g} = \pm\frac{p_i\mathbf{e}}{q_i g} \tag{2.19}$$

In (2.19), if $\nu_i = 1/3$ then the denominator is $3g$. The divisor of the fractional charge is equivalent to the multi-poles, and the dividend of the fractional charge is the multi-charges. The filling factor ν_i is a variable of the charge-pole ratio. Under extremely low temperature, the Ga-As-Al nucleus in I/FQHE must respond to a high magnetic field.

2-4 Self-generation of Filling Factors

The filling factors ν_i obtained from many experiments of the IQHE and FQHE are

$$\begin{aligned} \nu_I &= n, \cdots 8, 7, 6, 5, 4, 3, 2, 1 \\ \nu_F &= \frac{1}{3}, \frac{2}{5}\cdot\cdot\frac{2}{3}, \frac{3}{5}\cdot\cdot\frac{p}{2mp\pm 1}\cdot\cdot\frac{1}{2}, \frac{1}{4}\cdot\cdot\frac{4}{11}, \frac{6}{13}\cdot\cdot\frac{p_i}{q_i} \end{aligned} \tag{2.20}$$

where n, m, and p are all integers, and p_i/q_i is a fraction. Current FQHE theories are all based on the *arbitrary* fractional filling factors but not self-generated.

Next, let's see if the filling factor ν_i can be self-generated by the math formula of $\alpha^{1/2}$ in (2.9). Since $\mathbf{e} = (\underline{2\sigma_0 h\alpha})^{1/2}$, the *fractional* charge $|\pm\mathbf{e}_\nu| = \nu_i|\pm\mathbf{e}|$ is proportional to a *fractional* fine structure constant $|\alpha_\nu^{1/2}| = \nu_i|\alpha^{1/2}|$. Both ν_i and α are dimensionless numbers. Modifying the Wallis products

(1656) for the $\frac{\pi}{2}$ in (2.17)[75]

$$\begin{aligned}\frac{\pi}{2} &= \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\frac{10}{9}\frac{10}{11}\cdots = \prod_{p=1}^{\infty} 1^2 4\left(\frac{p}{2p-1}\right)\left(\frac{p}{2p+1}\right) \qquad (2.21)\\ &= \left(\frac{2}{1}\frac{2}{3}\odot\frac{6}{5}\frac{6}{7}\odot\frac{10}{9}\frac{10}{11}\odot\cdots\right)\cdot\prod_{p=1}^{\infty} 2^2 4\left(\frac{p}{2\cdot 2p-1}\right)\left(\frac{p}{2\cdot 2p+1}\right)\\ &= \left(\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\odot\frac{8}{7}\frac{8}{9}\frac{10}{9}\frac{10}{11}\odot\cdots\right)\cdot\prod_{p=1}^{\infty} 3^2 4\left(\frac{p}{2\cdot 3p-1}\right)\left(\frac{p}{2\cdot 3p+1}\right)\\ &= \prod_{n,m,p=1}^{\infty} n^2 4\left(\frac{p}{2mp-1}\right)\left(\frac{p}{2mp+1}\right)\end{aligned}$$

where $\odot$ shows where the twin-fractions are taken out, and n may or may not be equal to m. From (2.9)

$$\begin{aligned}\alpha^{-1/2} &= \frac{K\Phi\pi}{M} = \frac{2K\Phi}{M}\prod_{p=1}^{\infty} n^2 4\cdot\left(\frac{p}{2mp-1}\right)\left(\frac{p}{2mp+1}\right) \qquad (2.22)\\ &= \left(\frac{p}{2mp\pm 1}\right)\left[\frac{2K\Phi}{M}\prod_{p=1}^{\infty} n^2 4\cdots\right]\\ &= \left(\frac{p}{2mp\pm 1}\right)\left[\alpha_\nu^{-1/2}\right] = \nu_i\cdot\alpha_\nu^{-1/2}\end{aligned}$$

or taking out n in (2.22) as the integer n in (2.20), it yields

$$|\alpha_\nu^{1/2}| = \begin{pmatrix} n & \frac{p}{2mp+1} \\ \frac{p}{2mp-1} & n \end{pmatrix}\cdot\left|\alpha^{1/2}\right| = \nu_i\cdot\left|\alpha^{1/2}\right| \qquad (2.23)$$

From (2.17), (2.18) and (2.23), 2D Hall conductivity σ_{xy} is linked to the vacuum conductivity σ_0 by the filling factors ν_i and the fine structure constant α[65]

$$\sigma_{xy} = \pm 2\alpha^{1/2}\alpha_\nu^{1/2}\varepsilon_0 c = \pm 2\nu_i\alpha\sigma_0 \qquad (2.24)$$

i.e., $R_{xy} = \pm Z_0/2\alpha_\nu^{1/2}\alpha^{1/2} = \pm Z_0/2\nu_i\alpha$ where $Z_0 = \mu_0 c = 2\alpha h/\mathbf{e}^2 = 376.730313461\,[\Omega]$. Therefore, a good FQHE theory must expose the details

of the α information. This gives a clue for derivation of the fine structure constant math formula.

Why does the FQHE only choose $4\,[p/(2mp-1)]\,[p/(2mp+1)]$ or $4nn$ in the Wallis formula? This is because a complex product for the real part of two complexes (z_1, z_2) is equal to $4\mathrm{Re}(z_1)\mathrm{Re}(z_2) = (z_1 + \check{z}_1)(z_2 + \check{z}_2)$, where $z_1 = (x_1 + iy_1)e^{i\omega t}$ and $\check{z}_1 = (x_1 - iy_1)e^{-i\omega t}$ are the complex conjugates. It needs $4nn$ or $4\,[p/(2mp-1)]\,[p/(2mp+1)]$ in (2.22). According to Laughlin, the wave function for a filling factor $\nu_i = 1/\mathrm{M}$ is[70]

$$\Psi_{\mathrm{M}}(z_1 \cdots z_N) = \prod_{j\langle k\rangle}^{N} (z_j - z_k)^{\mathrm{M}} \exp\left(-\frac{1}{4}\frac{1}{l_0^2}\sum_{j}^{N}|z_j|^2\right) \tag{2.25}$$

where $z_j = x_j + iy_j$ is a complex number representing the position of the j^{th} particle on the plane, $l_0 = \sqrt{\hbar/\mathbf{eB}} = \sqrt{h/2\pi\mathbf{eB}}$ is the magnetic length, and M is an *odd* integer to satisfy the Pauli principle. 4 is also in the exponential term in (2.25).

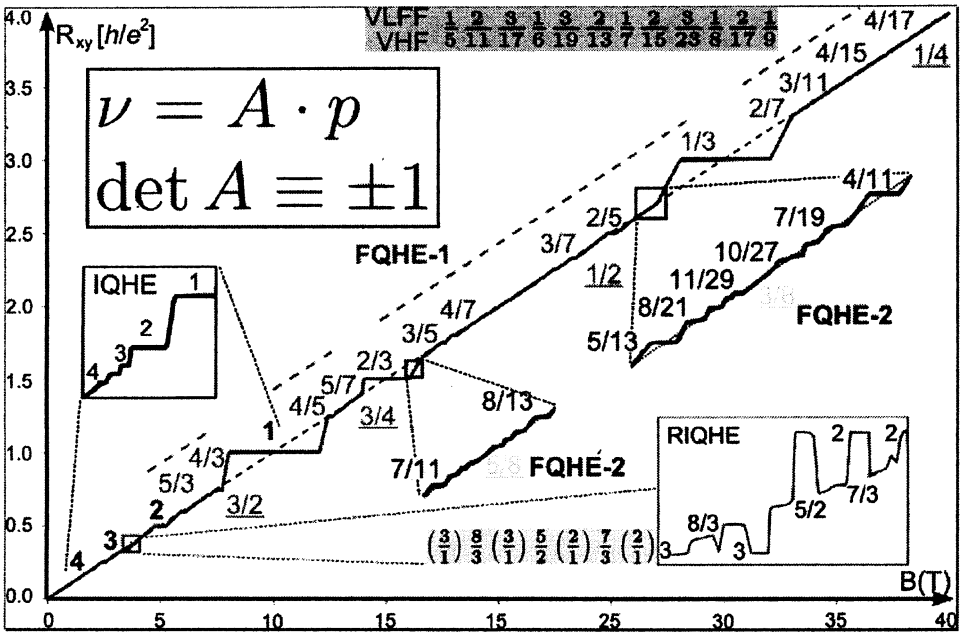

Fig. 2.3: I/FQHE resistance plateaus measured in GaAs-GaAlAs with a perpendicular magnetic field **B** [T]. The enlargement shows the FQHE-2, IQHE, RIQHE, VLFF-VLF.

In **Fig. 2.3**, we have $R_{xy} = 1/\nu$ in $[h/\mathbf{e}^2]$ unit, where the filling factor ν_i is arranged from largest to smallest, when $\mathbf{B}$ is increased from 0 to 40 [T]. These filling factors ν_i are a summation of many experimental reports, which are grouped as IQHE, RIQHE, FQHE-1, FQHE-2, Very Low Filling Factors in Very High Field, the even-denominator fraction, etc. It is the fingerprint of the fine structure constant.

Charge coupling combines the IQHE and FQHE, i.e., the algebraic sum of ν_I and ν_F in (2.20) and (2.23)

$$\nu_i = \nu_{n,m,p} = \underbrace{\pm n}_{\text{IQHE}} \underbrace{\pm \frac{p}{2m \cdot p \pm 1}}_{\text{FQHE}} = \underbrace{\frac{(\pm 2nm \pm 1) \cdot p \pm n}{2m \cdot p \pm 1}}_{\text{FQHE-1}} \Rightarrow \underbrace{\frac{\boldsymbol{a} \cdot p \pm \boldsymbol{b}}{\boldsymbol{c} \cdot p \pm \boldsymbol{d}}}_{\text{FQHE-2}} = \mathbf{A} \cdot p \tag{2.26}$$

where $n = 0, 1, 2 \ldots$, $m = 1, 2, 3 \ldots$, and $p = 1, 2, 3 \ldots$; $\boldsymbol{a} = \pm 2nm \pm 1$, $\boldsymbol{b} = n$, $\boldsymbol{c} = 2m \cdot p$ and $\boldsymbol{d} = 1$. Next, we give some ν_i examples of (2.26) to compare with **Fig. 2.2** and **Fig. 2.3**.

Let $n = m = +1$ and $p = 1, 2, 3 \ldots$ (the limits are $\frac{3}{2}$ and $\frac{1}{2}$)

$$\begin{aligned}
\nu^{1,1}_{+-} &= 2, \frac{5}{3}, \frac{8}{5} \cdots \left(\frac{3p-1}{2p-1}\right) \cdots && [5 \to 7 \text{ [T]}] \qquad (2.27)\\
\nu^{1,1}_{++} &= \frac{4}{3}, \frac{7}{5}, \frac{10}{7} \cdots \left(\frac{3p+1}{2p+1}\right) \cdots && [7 \leftarrow 10 \text{ [T]}]\\
\nu^{1,1}_{-+} &= \frac{2}{3}, \frac{3}{5}, \frac{4}{7} \cdots \left(\frac{p+1}{2p+1}\right) \cdots && [15 \to 20 \text{ [T]}]\\
\nu^{1,1}_{--} &= \underline{0}, \frac{1}{3}, \frac{2}{5} \cdots \left(\frac{p-1}{2p-1}\right) \cdots && [20 \leftarrow 30 \text{ [T]}]
\end{aligned}$$

Let $n = 2$, $m = 1$, and $p = 1, 2, 3 \ldots$ gives the RIQHE region between

$\nu = 2-3$ (the limits are $\frac{5}{2}$ and $\frac{3}{2}$, at $4 \to 10$ [T])

$$\begin{aligned}
\nu^{2,1}_{+-} &= 3, \frac{8}{3}, \frac{13}{5} \cdots \left(\frac{5p-2}{2p-1}\right) \cdots \\
\nu^{2,1}_{++} &= \frac{7}{3}, \frac{12}{5}, \frac{17}{7} \cdots \left(\frac{5p+2}{2p+1}\right) \cdots \\
\nu^{2,1}_{-+} &= \frac{5}{3}, \frac{8}{5}, \frac{11}{7} \cdots \left(\frac{3p+2}{2p+1}\right) \cdots \\
\nu^{2,1}_{--} &= 1, \frac{4}{3}, \frac{7}{5} \cdots \left(\frac{3p-2}{2p-1}\right) \cdots
\end{aligned} \tag{2.28}$$

Let $n = 0$ and $m = 2$, with a limit of $\frac{1}{4}$ ($30 \to 40$ [T])[76]

$$\begin{aligned}
\nu^{0,2}_{-} &= \frac{1}{3}, \frac{2}{7}, \frac{3}{11}, \frac{4}{15} \cdots \left(\frac{p}{4p-1}\right) \cdots \\
\nu^{0,2}_{+} &= \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \frac{4}{17} \cdots \left(\frac{p}{4p+1}\right) \cdots
\end{aligned} \tag{2.29}$$

Let $n = 1$ and $m = 2$, with a limit of $\frac{3}{4}$ ($10 \to 15$ [T])

$$\begin{aligned}
\nu^{1,2}_{++} &= 1, \frac{4}{5}, \frac{7}{9} \cdots \left(\frac{3p+1}{4p+1}\right) \cdots \\
\nu^{1,2}_{--} &= \frac{2}{3}, \frac{5}{7}, \frac{8}{11} \cdots \left(\frac{3p-1}{4p-1}\right) \cdots
\end{aligned} \tag{2.30}$$

For both the IQHE and FQHE, Hall resistance usually increases when filling factor ν_i decreases. RIQHE at $n > 1$ shows the alternating switch between the IQHE and FQHE.[77] From (2.27) to (2.30), all *neighboring* fractions obey a transition rule $|p_i q_{i+1} - p_{p+1} q_i| = 1$ (except 0, it is also invalid between the limitations).

The *even*-denominator in the FQHE should not really exist if it is due to limitations (e.g., $5 \times 10^8/10^9 + 1) \approx \frac{1}{2} \neq \frac{1}{2}$).[77] However, the even-denominator fraction occurs in RIQHE and FQHE,[8] and works with *neighboring* ν_i as $|ad - cb| = 1$ (e.g., $\frac{1}{2}$ with $\frac{1}{3}$ or $\frac{2}{3}$ will have $3 - 2 = 4 - 3 = 1$);

[8] $\overline{\nu}_{(2n\pm1)/2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \cdots$; $\overline{\nu}_{(2n\pm1)/4} = \frac{1}{4}, \frac{3}{4}, \cdots \frac{17}{4}, \frac{19}{4} \cdots$

they are yielded by Euler's infinite product rearranged as same as (2.21) ~ (2.23).

$$\begin{aligned} \frac{\pi}{2} &= \frac{3}{2}\frac{5}{6}\frac{7}{6}\frac{11}{10}\frac{13}{14}\frac{17}{18}\frac{19}{18}\frac{23}{22}\cdots \qquad \left[\frac{all\,odd\,prime}{discrete\,even}\right] \\ \frac{\pi}{4} &= \frac{3}{4}\frac{5}{4}\frac{7}{8}\frac{11}{12}\frac{13}{12}\frac{17}{16}\frac{19}{20}\frac{23}{24}\cdots = \left(\frac{1}{2}\frac{3}{2}\right)\left(\frac{1}{2}\frac{5}{2}\right)\left(\frac{1}{4}\frac{7}{2}\right)\left(\frac{11}{2}\frac{1}{6}\right)\cdots \end{aligned} \tag{2.31}$$

The *even*-denominators also appear in graphene, and may work for fault-tolerant quantum computations.[78] In fact, each of the *even*-denominator fractions are linked to multi-pairs of approximate *boson*-like states, where $R_{xx} \neq 0$ to form the wide bridge on the H-shaped minima. Due to this effect, $R_{xx} \neq 0$ may occur when $B = 0$ in the IQHE. They can be photon-induced to ZRS in a sample with an ultra-high mobility ($\mu > 10^7$ $[cm^2/\mathrm{Vs}]$) or with a long free path ($\lambda > 300\mu\mathrm{m}$), which is also the necessary condition for the FQHE.[79, 80]

It is interesting that the pair ($\frac{1}{3}$, $\frac{2}{5}$) in (2.27) creates a secondary fraction and generates several higher order filling factors ($\frac{4}{11}$ and $\frac{5}{13}$).[81] It can be $\frac{3p\pm1}{8p\pm3}$ or $\frac{3p\pm2}{8p\pm5}$, both produce a limit of $\frac{3}{8}$.[82] The pair ($\frac{2}{3}$, $\frac{3}{5}$) in (2.27) has fractions $\frac{7}{11}$ and $\frac{8}{13}$ produced by $\frac{5p\pm2}{8p\pm3}$ *or* $\frac{5p\pm3}{8p\pm5}$ with a limit $\frac{5}{8}$. We can get other secondary fractions in a similar way. Double solutions in Composite Fermion must have different physical construction for FQHE-2.[67] The interpretation of the *fractional* fine structure constant avoids this contradiction.

2-5 General Criterion of the Filling Factors

The geometrical Ford circles on the edge (**Fig. 2.4**(a)) follow the Pythagoras Theorem.[83] The general formula of all *neighboring* fractions a/c and b/d obeys

$$\nu = \frac{\boldsymbol{a}\cdot p \pm \boldsymbol{b}}{\boldsymbol{c}\cdot p \pm \boldsymbol{d}} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \cdot p = \mathbf{A}\cdot p \qquad (p = 0, 1, 2\ldots) \tag{2.32}$$

It yields a limit of a/c with the initial fraction b/d, where c is an even number; d is the odd-denominator of the initial *neighboring* fraction. a, b, c, and

d in the 2×2 matrix $\mathbf{A}$ must satisfy $\det \mathbf{A} = ad - cb = \pm 1$. The determinant of 2×2 Pauli metrics have the same criteria $\det(\sigma_i) = -1$. Equation (2.32) belongs to the special linear group $\mathrm{SL}(2, \mathrm{R})$ and $|ad - cb| \equiv 1$ is an universal rule wherever the $\pi/2$ fraction occurs. The quantization by using the quantum group $\mathrm{SL}(2, \mathrm{R})$, the moduli space of $SU(2)$ connections $(\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1})$ on the 2-dimensional *torus*, Möbius transformation, Continued Fraction, Weyl quantization and Lie group $\mathbf{GL}_2(\mathbb{R})$ are all mathematically connected.[84] In the continued model, $y = (ax \pm b)/(cx \pm d)$ is an equilateral hyperbola as shown in **Fig. 2.4**(b), with the rectangular center point at $(-d/c,\ a/c)$.

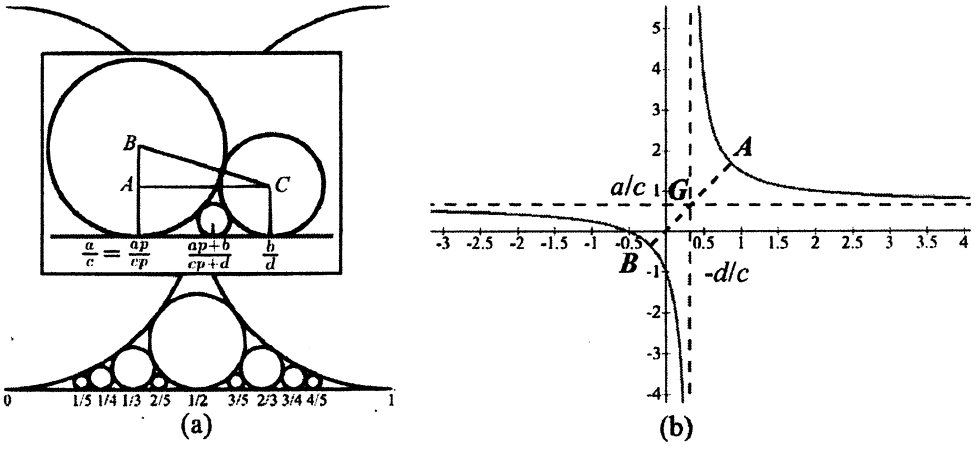

Fig. 2.4: (a) Ford circles for describing the gaskets on edge. Enlargement shows Pythagoras' Theorem. (b) The equilateral hyperbola for $y = (ax \pm b)\,/\,(cx \pm d)$ with a rectangular center G at $(-d/c,\ a/c)$.

VLFF-VHF and RIQHE states in **Fig. 2.3** are a good evidence of this transition rule $\det \mathbf{A} = ad - cb = |p_i q_{i+1} - p_{i+1} q_i| = \pm 1$.[85, 86, 87][9]

$$\nu_{RIQHE} = \left\{ \begin{matrix} \frac{4}{1} & \frac{\mathbf{19}}{\mathbf{5}} & \frac{4}{1} & \left(\frac{\mathbf{7}}{\mathbf{2}}\right) & \frac{3}{1} & \frac{\mathbf{16}}{\mathbf{5}} & \frac{3}{1} \\ \frac{3}{1} & \frac{\mathbf{8}}{\mathbf{3}} & \frac{3}{1} & \left(\frac{\mathbf{5}}{\mathbf{2}}\right) & \frac{2}{1} & \frac{\mathbf{7}}{\mathbf{3}} & \frac{2}{1} \end{matrix} \right. \tag{2.33}$$

[9] e.g., $4 \times 5 - 1 \times 19 = 20 - 19 = 1$. It is also valid in the quantum interference of the 2D electron collision.

Although a doped semiconductor is not a perfect crystal and may not have all the fractions in one sample, the 256 inequalities of experimental filling factors ν_i in many I/FQHE experiments show that $a/c > b/d$ satisfies

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb = \pm 1 \tag{2.34}$$

and there is a symmetric pattern of even-denominators in 256 ν_i, which is graphically illustrated in **Fig. 2.5** on the order from the highest to the lowest filling factor (i.e., $\nu_n > \nu_{n+1}$).[88][10]

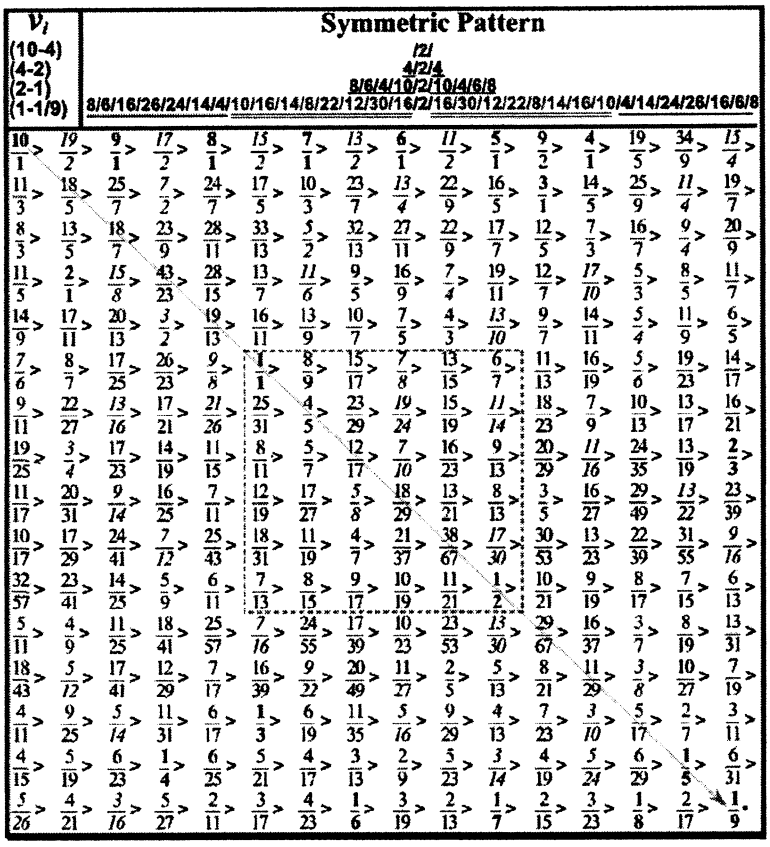

Fig. 2.5: The symmetric pattern of even-denominators in 256 ν_i

[10] Any new fractions can be easily added into the table according to (2.25), e.g., possibly 19/8 between 12/5 and 7/3.

This is the reality, behind all of the confusing fractions of **I/FQHE**, linked to Euler's convergence improvement transformation of $\pi/2$ [11]

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!!} = 1+\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}+\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\cdot\frac{4}{9}\cdots \tag{2.35}$$

$$= \sum_{n=0}^{\infty} \frac{n!\,[2^n n!]}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{n!(2n)!!}{(2n+1)!} = 1+\frac{1}{3}\Big(1+\frac{2}{5}\Big(1+\frac{3}{7}\Big(1+\frac{4}{9}\Big(1+\cdots\Big)\Big)\Big)\Big)$$

where all neighboring fractions obey $|ad-cb| \equiv 1$ (e.g., $2\times 3-1\times 5=1$).

Without a doubt, the filling factor ν_i of 2D fractional charge comes from the 2D geometrical constant π, due to the nucleon electric field of spherical symmetry being broken by an external magnetic field. However, the π in $\nu = 2\pi l_0^2 N$ is canceled by the π in $l_0^2 = \hbar/\mathbf{eB} = h/2\pi\mathbf{eB}$ that has no donation to ν_i, and there is no π in $\sigma_{xy} = \nu\mathbf{e}^2/h$ or $\mathbf{e} = [2\alpha\varepsilon_0 hc]^{1/2} = [2\alpha h/\mu_0 c]^{1/2} = [2\alpha\sigma_0 h]^{1/2} = [2\alpha h/Z_0]^{1/2}$. **The I/FQHE experiment clearly shows that the $\alpha^{1/2}$ formula must contain $\log e^{\pm 1}$ for $\pm$ sign and $1/\pi$ for ν_i. If there is $\log e^{\pm 1}$ and $1/\pi$ in the $\alpha^{1/2}$ formula, from $\frac{\log e^{\pm 1}}{\pi\alpha^{1/2}} = 1.618273$, one can easily get ϕ or Φ, then get $K \cong 1$.** This is a strong authorization of (2.9), a simple physical explanation of I/FQHE, beyond the current unsatisfactory theories.

Unlike α being the interaction of two-charges, $\alpha^{1/2}$ is the ratio of elementary charge $\mathbf{e}$ [esu] and Planck charge $\mathbf{q_P} = \sqrt{\hbar c}$ for a single-charge quantization. It has *one* value but *many* formats, $\phi - e - \pi$ each appear only once naturally in (2.9). Theoretically, the information entropy requires $\log e$ instead of $\log\Phi$ or $\log\pi$, so nature takes the minimum from $\underline{e^{1/\Phi\pi}} < e^{\Phi/\pi} < e^{\pi/\Phi} < e^{\Phi\pi}$ to build (2.9). The fact that so *many* applications involve $\phi - e - \pi$, however, prompted nature to adopt another *exclusive* number from the infinite natural numbers $(1, 2, \cdots, \infty)$ to pin-down K and the value of $\alpha^{1/2}$ in (2.9). Logically, an *exclusive* $K \cong 1$ should use the same math format as $\Phi - \phi - e - \pi$ in (2.10). From $\alpha^{-1} \approx 137.036\cdots$, $K \approx 1.00014768\ldots$ as continued fraction

$$K(J) = 1+\frac{1}{J+}\frac{1}{J+}\frac{1}{J+}\cdots = [1, \overline{6771}] \tag{2.36}$$

[11] $2^n n! = (2n)!! = 2\cdot 4\cdot 6\cdots(2n)$ *even product*, $(2n+1)!! = 1\cdot 3\cdot 5\cdots(2n+1)$ *odd product*

where $J = 3 \times 37 \times 61 = 6771$ is the "***the mysterious numbers***" for $\alpha^{1/2}$ assumed by Feynman (see 1-6).[52]

2-6 Sphenic Number and Double Factorial

The *true* α must be a theoretical math solution because even the best experimental data can not answer the simplest question: *why did nature choose this number instead of other numbers*? In order to determine the theoretical value of α, one needs to work on the J of $K(J)$ in (2.36). The sphenic number $J = 6771 = 3 \times 37 \times 61$ is experimentally safeguarded by 137.03599 ± 0.000125 ($137.035865 < \alpha^{-1} < 137.036115$). Möbius function $\mu(n) = \sum_{1 \le k \le n}^{\gcd(k,n)=1} e^{i2\pi k/n}$ is the sum of the primitive n^{th} roots of unity, with $\mu(3) = \mu(37) = \mu(61) = -1$ and $\mu(6771) = (-1)^3 = -1$. A set of sphenic numbers {1, **3**, **37**, **61**, 111, 183, 2257, **6771**} obeys

$$\{1, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, pq, pr, qr, \boldsymbol{pqr}\} \tag{2.37}$$

where a sphenic number $J = \boldsymbol{p} \times \boldsymbol{q} \times \boldsymbol{r}$. (2.37) is similar to the standard basis of geometric algebra in (2.1). Elementary charge is a special relativity invariant and independent of time. However, where do these 3, 37, 61 come from? How can we link them to the quantum theory?

The *infinite* prime double factorial equation

$$\mathbf{P(p)} = \frac{2^{(\mathbf{p}-1)}(\mathbf{p}-1)! + 1}{\mathbf{p}} = \frac{2(\mathbf{p}-1)!! + 1}{\mathbf{p}} \tag{2.38}$$

only has three solutions as prime sets $\mathbf{P}(\mathbf{p}) = \{2, \mathbf{3}, \mathbf{6.9 \times 10^{50}}, \mathbf{1.5 \times 10^{98}}\}$[12] when $\mathbf{p} = \{1, \mathbf{3}, \mathbf{37}, \mathbf{61}\}$[13]. (2.38) covers the entire prime set $(2, 3, 5, 7, 11 \cdots, \infty)$. The prime double factorial set $\{A\} = 2^{(\mathbf{p}-1)}(\mathbf{p}-1)! = [2(\mathbf{p}-1)]!!$ is a subset of set $\{B\} = 2^n n! = (2n)!!$ ($\{A\} \subseteq \{B\}$). $2^n n! = (2n)!!$ is equal

[12] $\mathbf{P}(37) = 6.9 \times 10^{50} = 690896939629347629014331483828706966091078572972973$

$\mathbf{P}(61) = 1.5 \times 10^{98} =$

$157269589866163720571999015535562658773483441682104752399411380250726154024406751104419672131147541$

[13] M. Bouayoun, http://oeis.org/A091824 (2004). Note 1 is not prime,

to the total permutation $\Gamma(n+1) = n!$ times the sum of k-combination 2^n (binomial coefficient of Pascal's triangle)[14]

$$\begin{aligned} 2^n n! &= (2n)!! = 2 \cdot 4 \cdot 6 \cdots 2n \\ &= P_n \sum_{k=0}^{n} C_k^n = P_n \sum_{k=0}^{n} \binom{n}{k} \\ &= 2^n \Gamma(n+1) = 2^n n \Gamma(n) \end{aligned} \tag{2.39}$$

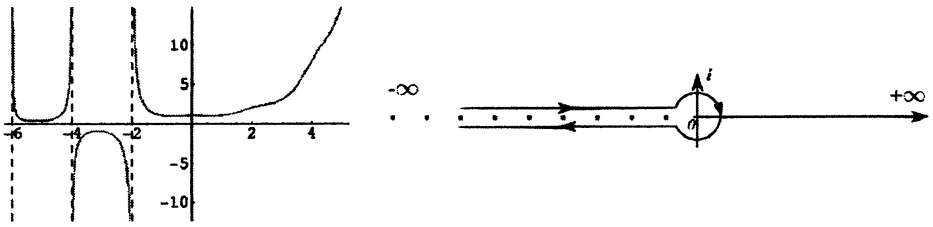

Fig. 2.6: Gamma Function and Hankel contour integral

Since it involves the Gamma function $\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}[(1+\frac{1}{n})^z/(1+\frac{z}{n})]$ $(z \neq -n)$ and $\Gamma(n+1) = n!$ in **Fig. 2.6**, the graphic pattern of $(2n)!!$ is similar to the $\Gamma(\frac{n}{2}+1)$ with poles at $-(2n+1) < -2$ $(n = 1, 2, \cdots)$. The *reciprocal Gamma* function is a Hankel contour-integral $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\oint_C e^t \cdot t^{-z} dt$ $(|\arg t| < \pi)$ starting from $t = -\infty$ and circling clockwise around the origin *0* before going back to $-\infty$ (similar to an electron absorbing and emitting a photon), and has a simple pole at $z = -n$ for every natural number and zero n; the residues are given by $\mathbf{Res}\,(\Gamma, -n) = (-1)^n/n\Gamma(n) = (-1)^n/n!$, and the sum of the residues of these poles equals

$$\sum_{n=0}^{\infty} \mathbf{Res}\,(\Gamma, -n) = \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = 1 - \frac{1}{1+}\frac{1}{2-}\frac{1}{3+}\frac{1}{2-}\frac{1}{5+}\frac{1}{2-}\frac{1}{7+}\cdots \tag{2.40}$$

$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n)} = \quad \ln e \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \ln\left(\frac{e^{\frac{1}{0!}}}{e^{\frac{1}{1!}}}\frac{e^{\frac{1}{2!}}}{e^{\frac{1}{3!}}}\frac{e^{\frac{1}{4!}}}{e^{\frac{1}{5!}}}\cdots\frac{e^{\frac{1}{(2n)!}}}{e^{\frac{1}{(2n+1)!}}}\cdots\right)$$

[14] $(2n)!! = \frac{(2n)!}{(2n-1)!!} = \frac{(2n+1)!}{(2n+1)!!} = (2n+1)!! \int_0^{\pi/2} \sin^{2n+1}\theta d\theta = 2^n \left(\frac{2}{\pi}\right)^{\sin^2(n\pi)/2} (n-1)!! n!!$

which is used as $\Gamma(1,+1) = 1/e = 0.367879\ldots$ for the negative charge electron in (2.13). There are no poles at $z > 0$, so $e = 2.71828\ldots$ does not exist (it also must be written as $e^{+1} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n)} = \ln(e^{\frac{1}{0!}} e^{\frac{1}{1!}} e^{\frac{1}{2!}} \cdots e^{\frac{1}{(2n)!}} e^{\frac{1}{(2n+1)!}} \cdots))$.[15] Asymmetry of the *Gamma* function may correspond to the absence of Positrons in nature. For the natural numbers $n > 0$, $(2n)!!$ yields $\{2, \underline{8}, 48, \underline{384}, 3840, \underline{46080}, 645120, 10321920, 1.86 \times 10^8, \underline{3.72 \times 10^9}, \ldots\}$, where the subset of $[2(\mathbf{p}-1)]!!$ is shown underlined.

Binomial distributions for $p = q = \frac{1}{2}$ involve $(2k)!!$ and $[2(n-k)]!!$

$$P(k) = \frac{n!}{k!(n-k)!} p^k q^{(n-k)} = \frac{n!}{(2k)!![2(n-k)]!!} \tag{2.41}$$

$(2n)!! = 2^n n!$ appears in the spherical Bessel function for electromagnetic waves, and in the Hermite polynomials for the eigenfunctions of the quantum harmonic oscillator with an eigenvalue of $E_n = (\boldsymbol{n} + \frac{1}{2})\hbar\omega$, $(\boldsymbol{n} = 1, 2, 3\ldots)$

$$\psi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \tag{2.42}$$

The orthogonal Hermite polynomials have $H_n^{(n)}(x) = 2^n n!$ and

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \begin{cases} 0 & (m \neq n) \\ 2^n n! \sqrt{\pi} & (m = n) \end{cases} \tag{2.43}$$

$2^n n!$ also appears in the Rodrigues' formula as $2^l l!$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l \tag{2.44}$$

$$2^l l! = \left[\frac{d^l}{dx^l} \left(x^2 - 1\right)^l\right]_{x \to 1}$$

[15] The Fransén-Robinson Constant for the contentious Gamma function $F = \int_0^{\infty} \frac{dx}{\Gamma(x)} = 2.807772420$ does not equal $e = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n)} = 2.718281828$ for the discrete Gamma function, where $\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left\{\left(1 + \frac{x}{n}\right) e^{-x/n}\right\}$ and $F - e = \int_0^{\infty} \{e^x [\pi^2 + (\ln x)^2]\}^{-1} dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp(\pi \tan\theta - \exp(\pi \tan\theta)) d\theta = 0.08948841$. We get $\alpha^{-1/2}(m_Z^2) = \frac{\Phi\pi}{\log F} = 11.3372868$ and $\alpha^{-1}(m_Z^2) = (\frac{\Phi\pi}{\log F})^2 = 128.53407$ at high energy scale.

which is linked to the associated Legendre polynomials in spherical harmonic expansion, to solve the Schrödinger equation (yielding $\boldsymbol{l} = 0,\ 1,\ 2 \ldots (\boldsymbol{n}-1)$, and $\boldsymbol{m} = 0,\ \pm 1, \ldots \pm \boldsymbol{l}$)

$$\begin{aligned} P_l^{|m|}(x) &= \frac{(-1)^m}{2^l l!}\left(1-x^2\right)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x) \\ &= \frac{(-1)^m}{2^l l!}\left(1-x^2\right)^{|m|/2}\frac{d^{l+|m|}}{dx^{l+|m|}}\left(x^2-1\right)^l \end{aligned} \tag{2.45}$$

where $(-1)^m$ is the extrinsic parity of the spherical harmonics. The spherical harmonics are defined by

$$Y_l^m(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{2.46}$$

which obeys $Y_l^{-l}(\theta,\phi) = \frac{1}{2^l l!}\cdot\sqrt{\frac{2l+1}{4\pi}}\sin^l\theta e^{-il\phi}$, $Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\sin\theta)$ and $Y_l^{-m}(\theta,\phi) \equiv (-1)^m\overline{Y}_l^m(\theta,\phi)$. It is used for solving the Schrödinger equation for the atomic electron configuration (**Fig. 2.7**). It can be further extended with the Condon-Shortley phase as $Y_{lm} \equiv (-1)^m Y_l^m$, and the Spin-weighted spherical harmonics as ${}_sY_{lm} = \sqrt{\frac{(l\mp s)!}{(l\pm s)!}}\eth Y_{lm}$.

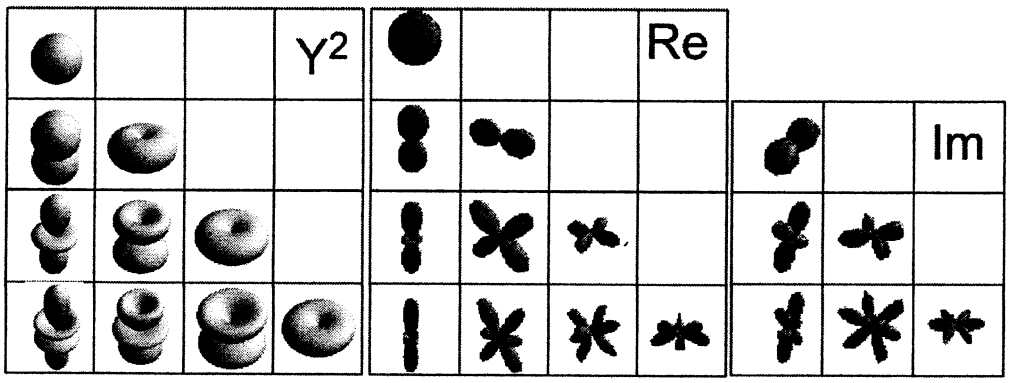

Fig. 2.7: The above illustrations show $|Y_l^m(\theta,\phi)|^2$ (*lift*), $\mathrm{Re}\,[Y_l^m(\theta,\phi)]^2$ (*middle*) and $\mathrm{Im}\,[Y_l^m(\theta,\phi)]^2$ (*right*).

Orthogonal polynomials were initially considered for continued fractions. Their inner products in Hilbert space obey $\langle \psi_n, \psi_m \rangle = \delta_{nm}$ and form the quantum numbers $(\boldsymbol{n}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{m_s})$. In the Path-Integral methods of QED, according to Weinberg, $2^n n!$ is also used in the Gaussian Multiple Integrals. Dyson S-matrix is sum of $(2^n n!)$ products, all of them are made up by the fine structure constant in various order.

In reality, they all involve spherical symmetry. The $\pi/2$ formula of Vieta in 1593 and Wallis in 1656 was linked to $2^n n! = (2n)!!$, and to all odd primes $\{3, 5, 7, 11, \ldots\}$ by Euler in 1737, with the countless variations

$$
\begin{aligned}
\frac{\pi}{2} &= \frac{2}{\sqrt{2}}\frac{2}{\sqrt{2+\sqrt{2}}}\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots && (Vieta-1593) \\
&= \lim_{n\to\infty} 2^n \sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} \\
\frac{\pi}{2} &= \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\frac{10}{9}\frac{10}{11}\cdots && (Wallis-1656) \\
&= \left[\frac{(2n)!!}{(2n+1)!!}\right]^2 = \left[\frac{2^n n!}{(2n+1)!!}\right]^2 && \left\{\frac{even}{odd}\right\}^2 \\
&= \prod_{n=1}^{\infty} \frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \prod_{n,m,p=1}^{\infty} 4n^2 \frac{\boldsymbol{p}}{\boldsymbol{2mp-1}}\cdot\frac{\boldsymbol{p}}{\boldsymbol{2mp+1}} \\
\frac{\pi}{2} &= \frac{3}{2}\frac{5}{6}\frac{7}{6}\frac{11}{10}\frac{13}{14}\frac{17}{18}\frac{19}{18}\frac{23}{22}\cdots && (Euler-1737) \\
&= \prod_{n=2}^{\infty} \frac{\mathbf{p}_n}{\mathbf{p}_n + (-1)^{(\mathbf{p}_n-1)/2}} && \left\{\frac{odd\,prime}{even}\right\}
\end{aligned}
\tag{2.47}
$$

where n may or may not equal to m. The infinite product of Vieta[16] make the elementary charge $\mathbf{e} = \sqrt{\sigma_0 h}\sqrt{2\alpha} = \sqrt{\sigma_0 h}\frac{\log e}{K\Phi}\sqrt{s}\sqrt{s+s\sqrt{s}}\sqrt{s+s\sqrt{s+s\sqrt{s}}}\cdots$, and Wallis has a consecutive sequence whereas Euler's has a discrete sequence

16

$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots = \sqrt{s}\sqrt{s+s\sqrt{s}}\sqrt{s+s\sqrt{s+s\sqrt{s}}}\cdots$ Vieta (1593)

in (2.47). In this way, Quantum theory does indirectly involve the discrete prime sequence.

$2^n n!$ also appears in a n-dimensional complex spheres, $V(D_n) = 2\pi^n/2^n n!$ $= 2\pi^n/(2n)!!$, [16] and still exists in the power series of the complete elliptic integral of the first and second kind. The spherical harmonic field gradually stretches into an elliptic and linear field if a particle moves at very high-speed or light-speed c: ○ ⋯ ☋ ⋯ ⟷. [17] $2^n n!$ disappears in light-speed, therefore, a photon has neither charge nor mass. Spherical harmonics are fundamental in physics theories and applications, such as the electron configuration, gravitation or electromagnetic field theories.

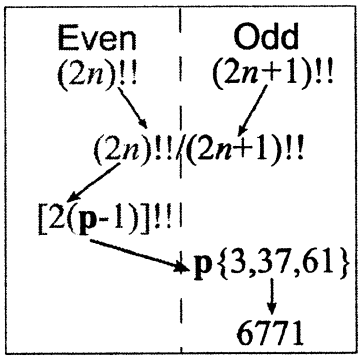

Fig. 2.8: In Nature, *odd* and *even* numbers are equally distributed, which makes π as the coupling constant of *odd* and *even* numbers in (2.47), i.e., $2(2\cdot4\cdot6\cdots)^2 = \pi(1\cdot3\cdot5\cdots)^2$ or prime~even $(2\cdot3\cdot5\cdot7\cdot11\cdot13\cdots) = \pi(2\cdot6\cdot6\cdot10\cdot14\cdots)$. The *even* number double factorial set $\{(2n)!!\}$ contains a subset $\{[2(\mathbf{p}-1)]!!\}$. The prime double factorial equation (2.38) yields a *odd prime* set {3, 37, 61}, making the *unique* sphenic number $(2^2-1^2)(4^3-3^3)(5^3-4^3) = 3\times37\times61 = 6771$ for the exclusive fine structure constant, where the *even/odd* numerical symmetry is broken.

[17] Special relativity will gradually distort the spherical harmonics. The relativistic Dirac equation can be solved by symmetric distortion, i.e., the wavefunction of *non*-Spherical harmonics which involves $2^l k!$ instead of $2^l l!$. However, charge only disappears at light speed, i.e., photon. The Dirac equation is for fermions running at very high-speed. In polar coordinates, a conic section $r = l/(1 \pm e\cos\theta)$ where $l = 0$ is a point (*origin O*), if $l > 0$ then varying $e \in \{0, (0 < e < 1), 1, (1 < e < \infty), \infty\}$ gives a circle ($e = 0$), ellipse ($0 < e < 1$), parabola ($e = 1$), hyperbola ($e > 1$) or line ($e \to \infty$).

Spherical harmonics involve $2^n n!$, and (2.38) with $2^{(\mathbf{p}-1)}(\mathbf{p}-1)!$ yields an *odd prime* set $\{3, 37, 61\}$ which produces the sphenic number 6771 (**Fig. 2.8**), where *even/odd* numerical symmetry is broken. $6771 = 3 \times 37 \times 61$ to a sphere is just like the stem on an apple. It denotes a unique point on an otherwise perfect sphere.

The uniqueness of the fine structure constant is based on this *even/odd* numerical symmetry breaking. Perhaps coincidentally, $\frac{37\times61}{2\times3} = 376.16\overline{6}$ is close to the value of vacuum impedance in (2.4).

Since the dimensionless α is exclusive in physics, is there a corresponding natural number in math? Here we show $J = 6771$ is the exclusive number for the exclusive fine-structure constant.

2-7 Variability of the Fine Structure Constant

We are not looking for a simple numeral solution but an explainable math-physical expression. The α variability has been discussed using two approaches: (A) In the zero-energy limitation for fermions, the value is fluctuated around $\alpha^{-1} = 137.036$; (B) In the high-energy running coupling as the electroweak unification, it appears in different values $\alpha^{-1}(m_Z^2) \sim 128.952$ for the effective fine structure constant. Let's first discuss (A).

(2.9) is a physical equation with the information units [ban], [hart], [nat]. In fact, $K = [1, \overline{6771}]$ in (2.36) can be a limitation of the π-form continued fraction $K(J, H) = 1+\frac{1^H}{J+}\frac{3^H}{J+}\frac{5^H}{J+}\cdots$ when $H = 0$. The value of $K^2(J, H)$ has the same pattern with α^{-1} ($\triangle K^2(J, H) = 6.1 \times 10^{-7}$ in **Fig. 2.9**). We can get the discrete experimental data by variation of H ($\triangle\alpha/\alpha = 6.1 \times 10^{-7}$ in **Fig. 2.9**). For example, let $H = \frac{2}{9}\pi(e^{+1} + e^{-1})^2 = 4\pi \sin^2\theta_w \cosh^2(\pm 1)$ then $\alpha^{-1}_{6.649} = \underline{137.03599907}395$ to match $\alpha^{-1}_{2010} = \underline{137.035999074}\,(44)$ (*inside the little green circle* in **Fig. 2.9**). Let $H = 6 + \gamma = 6.57721566$ then $\alpha^{-1}_{6.57721566} = \underline{137.035999173895}$ to match QED calculation $\alpha^{-1}_{2012} = \underline{137.035999174}\,(35)$. In this way, (2.9) gives the origin and the range of domain for α, including the values of CODATA.

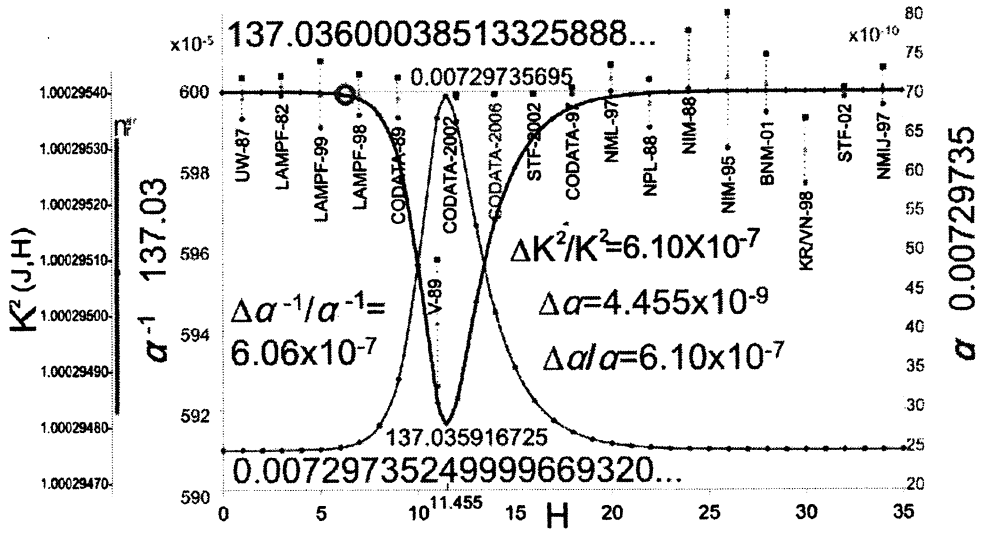

Fig. 2.9: Plot of $K^2(J,H) \sim n_r^{air}$, α^{-1} and α vs variable H with the compiled reports.

The values of CODATA come from selective reports based on the statistic average of discrete data.[48, 49] The CODATA recommended values constantly change and never exclude other experimental data. There are many real experiments yielding slightly different values of $\alpha^{-1} = K^2(J,H) \cdot \underline{\Phi^2\pi^2 \ln^2 10} = \frac{c}{v_e}$, where v_e is the velocity of the electron in the first "circular" Bohr orbit. Unfortunately, no report has been given on the reasons for differences arising in controlled experimental conditions.

As we discussed in the section of (2-1), **the fine structure constant is the coupling constant of two zones (electron/space).** In the determination of the elementary charge, the early Millikan oil-drop experiment involved Stokes law with the viscosity of *air*.[90] In **Fig. 2.9**, the variable region of $K^2(J,H)$ is approximately equal to the refractive index of air $n_r^{air} = \frac{c}{v_p} = (\varepsilon_r^{air} \cdot \mu_r^{air})^{1/2} = 1.00029507 \pm 2.4985 \times 10^{-7}$ (red-bar in **Fig. 2.9**), where $\varepsilon_r^{air} = 1.00058986 \pm 0.00000050$ and $\mu_r^{air} = 1.00000037$ are the relative permittivity and permeability of air ($N_2(78.084\%) + O_2(20.946\%) + \cdots$ in e-e covalent bond).[91] Physically, the effective $K(J,H) \sim n_r^{1/2} = (\mu_r^{e^2} \cdot \varepsilon_r^{e^2})^{1/4}$ is for the *single* electron. There is a single peak at $0 \leq H \lesssim \alpha^{-1/2} = 11.706$ in **Fig. 2.9**. Therefore, a modified (2.9) can be linked to the *effective*

electric charge that has been widely discussed in the *low-energy* physical theory. According to Feynman, "*But no such ideal electrons exist. ... a real electron, which emits and absorbs its own photons from time to time, and therefore depends on the amplitude for coupling, J.*"[52] The refractive index is equal to the ratio of light-speed in a vacuum and medium. Charge quantization begins during the photon deceleration of light-speed. It is logical to think that the gas model of the *electron* in an atom breaks the vacuum space-time symmetry. We have proved that electrons are the charged oscillators in blackbody radiation.[13] We also shown that the fine structure constant is the corner-store in the quantum theory.[92] However, many factors can make the quantum fluctuation of the *macro*-refractive index, such as density (pressure, temperature, ...), impurity absorption (CO_2, H_2O, ...), and so on. The refractive index of a solid or liquid is more complicated as a tensor, which involves a disturbance of the field and the positions and velocities of the charged particles (electrons) within the material. It is noticed that the monatomic gas of $^{133}C_s^{6S1}$ with the *single* 6S1 electron in the photon-recoil measurement of atom interferometry yield $\alpha^{-1}(Atom\ Interferometry) = 137.0360003(10)$, which is close to the theoretical limitation in **Fig. 2.9**.[93] We need more precision *controlled* data for the *solo* photon-electron *micro*-refractive index.

2-8 The Electro-weak-strong Unification

The β^- decay ($\mathbf{n}^0 \rightarrow \mathbf{p}^+ + \mathbf{e}_L^- + \bar{\boldsymbol{\nu}}_e^R$) involves the weak interaction. The electroweak interactions have been merged by $SU(2)_L \otimes U(1)_Y$ with the coupling of charge $\mathbf{Q}$, isospin $\mathbf{I}_W$ and hyper-charge $\mathbf{Y}_W$, linked to the Fermi coupling constant $G_F \cong \sqrt{2}\mathbf{g}^2/8m_w^2$, by defining $\boldsymbol{e} = \mathbf{g}\sin\theta_w = \mathbf{g}'\cos\theta_w$ ($\mathbf{g} \sim W^{\pm}$ and $\mathbf{g}' \sim Z^0$),[18] and the Weinberg angle $\cos\theta_w = m_w/m_z$ or $\tan\theta_w = \mathbf{g}'/\mathbf{g}$ have to be measured experimentally.[96, 97, 98]

[18] $G_F = 1.166364(5) \times 10^{-5}[\mathrm{GeV}^{-2}]$, $\mathbf{g}$, $\mathbf{g}'$, $\alpha_{em} = \frac{e^2}{4\pi} = \frac{1}{137}$ and $e = 0.302822119$ are in the natural unit. The one-loop *beta*-function in QED is $\beta(e) = \frac{e^3}{12\pi^2}$ or $\beta(\alpha) = \frac{2\alpha^2}{3\pi}$, then $c_4\alpha^2 = \frac{\alpha^2}{3\pi^2} = \frac{\beta_1(\alpha)}{2\pi}$, it is *proportional* to the energy scale.

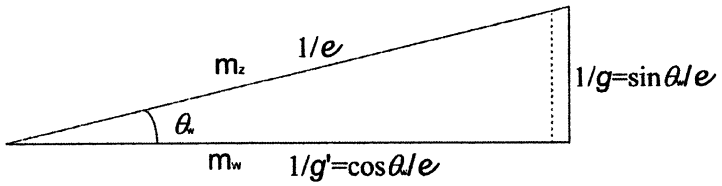

Fig. 2.10: The charge and mass relationship in the electroweak interactions $SU(2)_L \otimes U(1)_Y$

The anomalous electron will couple with the weak force in the nucleus as the *electroweak coupling* (per Pythagorean theorem in **Fig. 2.10**).

$$\begin{aligned} \eta &= \frac{\alpha}{\sin^2\theta_w} = \frac{\mathbf{g}^2}{4\pi} = \frac{9}{2}\alpha = \frac{\alpha}{0.2222\overline{2}} = \frac{1}{30.452} \\ \eta' &= \frac{\alpha}{\cos^2\theta_w} = \frac{\mathbf{g}'^2}{4\pi} = \frac{9}{7}\alpha = \frac{\alpha}{0.7777\overline{7}} = \frac{1}{106.583} \end{aligned} \tag{2.48}$$

So, $G_F \cong \eta\pi/\sqrt{2}m_{\rm W}^2 = 9\pi\alpha/\sqrt{8}m_{\rm W}^2$.

The effective fine structure constant at high energy level shows $\alpha^{-1}(m_z^2) \cong 128.952(49)$.[99] We get

$$\begin{aligned} \alpha^{1/2}(m_z^2) &= \frac{\pm\log F}{\Phi\pi}(1-\frac{2}{9}\alpha) = \frac{\pm\log F}{\Phi\pi}[1-\frac{2}{9}(\frac{\pm\log e}{K\Phi\pi})^2] \\ &= \pm\frac{\log F}{\Phi\pi}(1-\alpha\cdot\sin^2\theta_w) = \pm\frac{\log F}{\Phi\pi}\{1-(\frac{\pm\log e}{K\Phi\pi})^2[1-(\frac{m_{\rm w}}{m_z})^2]\} \\ &= \frac{\log F^{\pm 1}}{\Phi\pi}(1+Q_dQ_u\alpha) = \alpha_{z'}^{1/2}(1-\alpha\sin^2\theta_w) = \pm 0.0880614 \end{aligned} \tag{2.49}$$

or $\alpha^{-1}(m_z^2) = 128.9519596$, where Fransén-Robinson constant $F = \int_0^\infty \frac{dt}{\Gamma(t)} = \lim_{\lambda\to 0}\lambda E_{\lambda,0}(1) = \lim_{\lambda\to 0}\lambda\sum_{n=0}^{\infty}\frac{1}{\Gamma(\lambda n)} = 2.807772420$[19] as the continued gamma function integral for the time interval $\triangle t \to 0$, vs. $e = \sum_{n=0}^{\infty}\frac{1}{\Gamma(n)} = 2.71828$ as the

[19]The Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)} = \frac{1}{2\pi i}\oint_C \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}dt$ is the contour starts and ends at $-\infty$ and circles around the singularities of the integrand (**Fig. 2.6**), for the Continuous Time Random Walk-CTRW.[100]

infinite sum of the discrete gamma function (see the footnote 15). In (2.49) $\eta_z = \alpha_z/\sin^2\theta_w = 28.656$ and $\eta_z^{'} = \alpha_z/\cos^2\theta_w = 100.296$.[20] For Z boson, $\alpha^{1/2}(m_z^2) = \frac{\log F^{\pm 1}}{\Phi\pi}(1 - \alpha\sin^2\theta_w)$. It further approves the correctness of the fine structure constant formula based on $\alpha^{1/2} \approx \pm\frac{\log e}{\Phi\pi}$.

Euler constant $e = \lim\limits_{n\to 1} I_n$ and Fransén-Robinson constant $F = \lim\limits_{n\to\infty} I_n$ can be unified as the limitation of the Riemann sum I_n[101]

$$I_n = \frac{1}{n}\sum_{n=0}^{\infty}\frac{1}{\Gamma(\frac{k}{n})} = \begin{cases} e = \sum\limits_{n=0}^{\infty}\frac{1}{\Gamma(n)} = 2.718281828... & n = 1 \\ \frac{1}{\sqrt{4\pi}} + \frac{e}{2}\cdot\mathrm{erfc}(-1) = 2.7865848321 & n = 2 \\ \Downarrow \qquad\qquad\qquad\qquad \Downarrow & \\ F = \int_0^{\infty}\frac{dx}{\Gamma(x)} = 2.8077702420285... & n \to \infty \end{cases} \tag{2.50}$$

where the complementary error function

$$\begin{aligned} \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt = 1 - \mathrm{erf}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(2n-1)!!}{(2x^2)^n} \qquad (2.51) \\ = 1 - \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!}\cdots) \end{aligned}$$

or as a continued fraction

$$\begin{aligned} \mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt = 1 - \mathrm{erf}(x) \qquad (2.52) \\ &= \frac{xe^{-x^2}}{\sqrt{\pi}}\frac{a_1}{x^2+}\frac{a_2}{1+}\frac{a_3}{x^2+}\frac{a_4}{1+}\frac{a_5}{x^2+}\frac{a_6}{1+}\cdots \end{aligned}$$

where $a_1 = 1$, $a_m = \frac{m-1}{2}$ $(m \geq 2)$. The standard normal distribution, described by the probability density function $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, then the cumulative distribution function of the normal Gaussian distribution $\Phi(x)$ is

[20] GUT's expersions are $\alpha_1 = \frac{5}{3}\frac{\alpha}{\cos^2\theta_w} \sim \frac{1}{63.95}$; $\alpha_2 = \frac{\alpha}{\sin^2\theta_w} \sim \frac{1}{30.45}$; $\alpha_3 = \alpha_s \sim \frac{1}{8.54}$ and running.

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt = \frac{1}{2}\mathrm{erfc}(-\frac{x}{\sqrt{2}}) \tag{2.53}$$

This explains the statistic character of the electric charge at the high-energy scale.

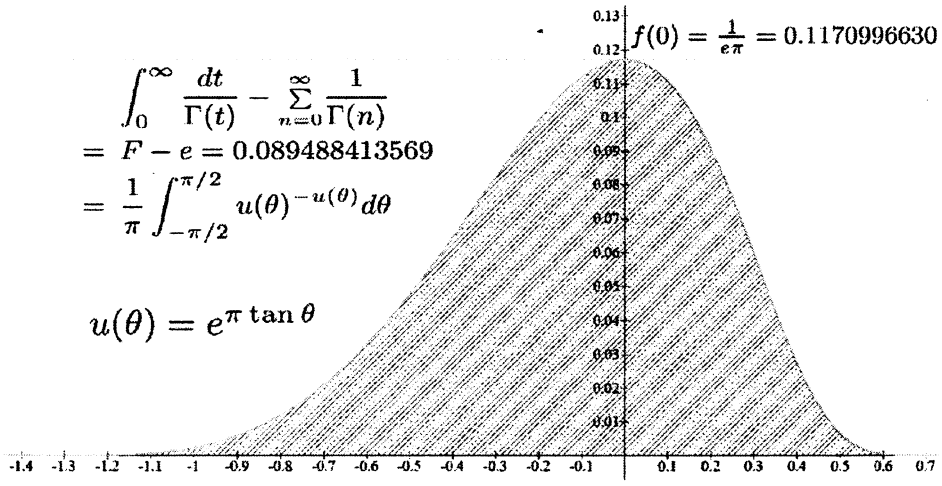

Fig. 2.11: $F - e$ integral has a maximum of $f(0) = \frac{1}{e\pi} = \frac{1}{4}(\frac{1}{2})^{3/2}(\frac{3}{2}\frac{3}{4})^{5/4}(\frac{5}{4}\frac{5}{6}\frac{7}{6}\frac{7}{8})^{9/8}\cdots = 0.1170996630$

Experimentally, the strong coupling constant $\alpha_s(m_Z^2) = 0.112 \sim 0.121$ and world average $\alpha_s(m_Z^2)_{1\,loop} = 0.1171 \sim \alpha_s(m_Z^2)_{3\,loop} = 0.1184$.[102] It is too early to tell whether it is related to $f(0) = \frac{1}{e\pi} = 0.117099$ of the $F - e$ integral in **Fig. 2.11**

$$F - e = \int_0^x \frac{dt}{\Gamma(t)} - \sum_{n=0}^{\infty} \frac{1}{\Gamma(n)} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} u(\theta)^{u(\theta)} d\theta = 0.08949488413569 \tag{2.54}$$

where $u(\theta) = e^{\pi \tan\theta}$

(2.50) can make alpha running from the starting points of SM or SUSY as **Fig. 2.12**[103, 104]

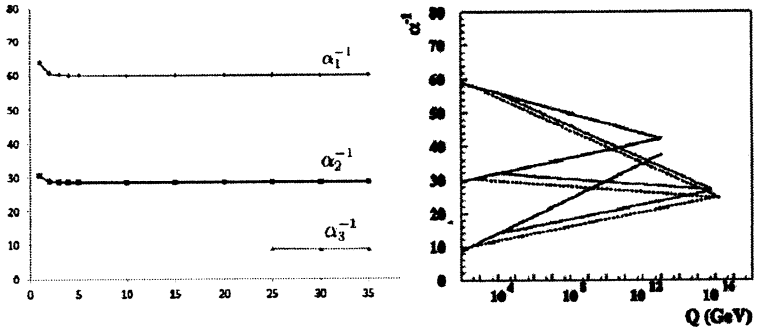

Fig. 2.12: $\alpha_1^{-1} = \frac{3}{5}\alpha^{-1}\cos^2\theta_w \sim 60$ and $\alpha_2^{-1} = \alpha^{-1}\sin^2\theta_w \sim 30$ calculated from (2.53) and $\alpha_3^{-1} = \alpha_s^{-1} \sim 8.5$ running from the starting points of SM or SUSY.

2-9 Information and Physical Reality

Feynman lectured that *"the observed coupling constant* ***e*** - the amplitude for ***a real electron to emit or absorb a real photon***. *It is* ***a simple number*** *that has been experimentally determined to be close to* -0.08542455. ... *a certain combination of* ***pi****'s and e's (the base of the natural logarithms), and* ***2****'s and* ***5****'s produces the mysterious coupling constant..."*[52]

Here we propose (2.9), alike Feynman's hunch, based on $\Phi - \phi - e - \pi$ in a modified *Euler* identity in (2.11) and $6771 = 3 \times 37 \times 61$ from (2.38), as a simple and physically understandable *unique* α math formula, which yields the mystery of *137* as

$$\begin{aligned}
\alpha^{-1} &\equiv (\pm K\Phi\pi \ln 10)^2 \equiv [\pm K \cdot 5 \cdot \ln 10 \cdot \prod_{n=1}^{\infty} \frac{(10n)^2}{(10n-1)(10n+1)}]^2 \qquad (2.55)\\
&= \{\pm[1,\overline{6771}] \cdot 10 \cdot \tanh^{-1}\left(\frac{10-1}{10+1}\right) \cdot \prod_{n=1}^{\infty} \frac{(10n)^2}{(10n-1)(10n+1)}\}^2\\
&= \{\pm[1,\overline{6771}] \cdot 2 \cdot 5 \cdot \tanh^{-1}\left(\frac{2\cdot 5-1}{2\cdot 5+1}\right) \cdot \prod_{n=1}^{\infty} \frac{(2\cdot 5n)^2}{(2\cdot 5n-1)(2\cdot 5n+1)}\}^2
\end{aligned}$$

where $\ln 10 = \ln 2 + \ln 5 = 2\tanh^{-1}\left(\frac{10-1}{10+1}\right)$, and the infinite product of $\Phi\pi$

is based on Euler's infinite product formula $\sin(\pi z) = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$, i.e.,[21]

$$\frac{\pi/k}{\sin(\pi/k)} \equiv \frac{k}{k-1}\frac{k}{k+1}\frac{2k}{2k-1}\frac{2k}{2k+1}\cdots = \prod_{n=1}^{\infty}\frac{(nk)^2}{(nk-1)(nk+1)} \quad (2.56)$$

Since $2\sin(\pi/10) = \phi$, let $k = 10$

$$\frac{2\cdot(\pi/10)}{2\sin(\pi/10)} \equiv \frac{\pi\Phi}{5} = \frac{10}{9}\frac{10}{11}\frac{20}{19}\frac{20}{21}\frac{30}{29}\frac{30}{31}\cdots = \prod_{n=1}^{\infty}\frac{(10n)^2}{(10n-1)(10n+1)} \quad (2.57)$$

The α is built with continued fractions or infinite products because Nature can't use the decimal $\pi = 3.14159\ldots$ as we do. She can only work with the ratio of integers after the physical quantity is counted as numbers of quanta. However, as an information entropy, the logarithmic format should be used, such as, $\log e^w$, $\ln 10^w$. Now we can understand the importance of an alternating series to the electron, and by extension explain the absence of positions in nature (i.e., (2.13)-(2.15) and (2.41)).

This number *137* serves as a bridge in the quantum (h) to electricity ($\mathbf{e}$), relativity (c), and vacuum (ε_0).[59] It is an *Universal information number* regulating many physical ratios, such as charge $\left(\frac{\mathbf{e}^2}{\hbar c}\right)$, velocity $\left(\frac{v_{\mathbf{e}}}{c}\right)$, energy $\left(2\pi\frac{E_{\mathbf{e}}}{E_\phi}\right)$, length $\left(\frac{r_{\mathbf{e}}}{\lambda\!\!\!^{-}_{\mathrm{c}}} = \frac{\lambda\!\!\!^{-}_{\mathrm{c}}}{a_0}\right)$, angular momentum $\left(\frac{\mathbf{e}^2/c}{\hbar} = \frac{\hbar}{m_e c a_0}\right)$, conductivity $\left(\frac{G_0}{4\sigma_0}\right)$, and impedance $\left(\frac{\mu_0 c}{\mathbf{e}^2/h}\right)$, *etc.* As we discussed in the section of (2-1), **the fine structure constant is a coupling ratio between the particle quanta properties and the surrounding spacetime properties, i.e., the coupling between wave and particles.** It frustratingly involves so many physical areas and holds so many physical mysteries. This paper only serves as an initial outlook into the mysteries related to the fine structure constant. This paper builds the basic math formula of α by studying I/FQHE, with countless variations of $\phi - e - \pi$ and the core of $J = 6771$ is based on $\{3, 37, 61\}$, by the principles of spherical symmetry

[21] The sinc-function $\mathrm{sinc}(\pi z) = \frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2}) = \frac{1}{z\Gamma(z)\Gamma(1-z)}$ have been used in many physical applications, such as, the double-slit interference. [92]

and asymmetry. The following study will investigate this α math formula in other applications. Much more work is needed in the future, including modification, confirmation, correction, or debating other solutions.[22]

All physical equations carry math relationships. At root, there must be a magic group of numbers with basic principles at work in nature, according to which the natural mysteries can be understood. We usually imagine circles when discussing π without realizing that it is a coupling of *even* and *odd* numbers; we think of beauty when talking about ϕ, and don't grasp its statistic meaning; we use e more often in engineering and economics, and don't understand it to be a coupling of boson and fermion; we fixate on *137*, and don't know the uniqueness of $J = 6771$ in the infinite natural numbers. We might be too influenced by our eyes, and poorer at understanding the real logical thinking in the brain. $\alpha^{-1/2}$ in (2.48) can be easily converted to the ellipsoid volume $V = \frac{4}{3}\pi abc$,[23] and $\Phi\pi$ can be linked to the golden circle. However, it can also be linked to the number $10 = 2 \times 5$.

We probably need to look at both the number theory and topology, and constantly check with the physical experiments. Otherwise, we may run into the trap of metaphysics. There are many α math formulas based on a geometric construction only, the logical analysis of the number theory has been neglected. If we construct a geometric α, the elementary charge and spin of the Fermions simply could not be the Lorentz invariant ($\mathbf{e} = \mathbf{e}_0$ and $\hbar/2 = \hbar_0/2$ v.s. $l = l_0/\gamma$ and $m = \gamma m_0$). We should not refuse to look at the number theory because it has no conventional physical unit. In fact, (2.9) has the information *units* in [ban], [hart], [nat].

Einstein was a pioneer on searching for an α math formula. He said: "*It is my conviction that pure mathematical construction enables us to discover the concept and the laws connecting them, which gives us the key to understanding of the phenomena of Nature.*"[94]

It is well-known that only 5 Fermat primes $\{3, 5, 17, 257, 65537\}$ satisfy $F_n = 2^{2^n} + 1$. Here we show the prime double factorial equation (2.38) only has 3 solutions $\{\mathbf{3}, \mathbf{37}, \mathbf{61}\}$, which initialize the fermion charge quantization

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[22] such as, $\alpha^{1/2} = \frac{e\pi}{2^2 5^2} = \frac{1}{5^2}\cdot\frac{e}{2}\cdot\frac{\pi}{2} = \frac{1}{25}\left(\frac{2}{1}\right)^{3/2}\left(\frac{2}{3}\cdot\frac{4}{3}\right)^{5/4}\left(\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\right)^{9/8}\left(\frac{8}{9}\frac{10}{9}\frac{10}{11}\frac{12}{11}\frac{12}{13}\frac{14}{13}\frac{14}{15}\frac{16}{15}\right)^{17/16}\cdots$, i.e., $\alpha = \frac{e^2\pi^2}{2^4 5^4} = \frac{1}{625}\left(\frac{2}{1}\right)^3\left(\frac{2}{3}\cdot\frac{4}{3}\right)^{5/2}\left(\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\right)^{9/4}\left(\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{14}{13}\cdot\frac{14}{15}\cdot\frac{16}{15}\right)^{17/8}\cdots$

[23] Let $a = \frac{3}{4}\ln 10$; $b = \Phi$; $c = K_0$; then $V_{ellipsoid} = \frac{4}{3}\pi abc = \alpha^{-1/2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$(K(3,37,61) \approx 1)$, $\alpha_{naked}^{1/2} = \log e/\Phi\pi = 1/\Phi\pi \ln 10$, associated by countless variations of $\Phi - \phi - e - \pi$ to constitute the Universe Information. We present the fine structure constants with the basic math constants, which is formulated as the beautiful Euler Identity $e^{\pm i\pi} + \Phi = \phi$. Many people put countless efforts on the fine structure constant. $\alpha^{1/2} \approx \pm\frac{\log e}{\Phi\pi}$ offers an another way to approach this most difficult issue of theoretical physics.

As Einstein opined, "*In a reasonable theory there are no (dimensionless) numbers whose values are only empirically determinable.*"[105] "*Dimensionless constants in the laws of nature, which from the purely logical point of view can just as well have different values, should not exist. To me, with my 'trust in God' this appears to be evident, but there will be few who are of the same opinion.*"[106]

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INITIATING SANTILLI'S ISO-MATHEMATICS TO TRIPLEX NUMBERS, FRACTALS AND INOPIN'S HOLOGRAPHIC RING: PRELIMINARY ASSESSMENT AND NEW LEMMAS

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Abstract

In a preliminary assessment, we begin to apply Santilli's iso-mathematics to triplex numbers, Euclidean triplex space, triplex fractals, and Inopin's 2-sphere holographic ring (HR) topology. In doing so, we successfully identify and define iso-triplex numbers for iso-fractal geometry in a Euclidean iso-triplex space that is iso-metrically equipped with an iso-2-sphere HR topology. As a result, we state a series of lemmas that aim to characterize these emerging iso-mathematical structures. These initial outcomes indicate that it may be feasible to engage this encoding framework to systematically attack a broad range of problems in the disciplines of science and mathematics, but a thorough, rigorous, and collaborative investigation should be in order to challenge, refine, upgrade, and implement these ideas.

Keywords: Geometry and topology; Iso-number; Triplex number; Iso-triplex number; Euclidean triplex space; Euclidean iso-triplex space; Inopin holographic ring; Iso-sphere holographic ring; Fractal; Iso-fractal; Mandelbrot set; Chaos theory.

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1 Introduction

A number is a mathematical object that is used to count, label, and measure. Number systems are fundamental to all quantitative sciences because they are used to encode the state space and transition space of experimental and observable features in nature. Such systems are tools that let scientists explore ideas and quantify experimental results. Historically, advances in science, engineering, and technology have pushed the mathematical definition of a number to include additional structures such as 0, negative, integer, rational, irrational, real, complex, and quaternion numbers in order to satisfy the imposed representational demands of these disciplines.

A number is an element, so a set of numbers is a set of elements, where the number set is equipped with addition and multiplication operators to establish a number field that complies with five number field axioms [1]. Let

$$X = \{x_1, x_2, x_3, ...\} \tag{1}$$

be a number field, where operators can be applied to numbers for addition $(+)$ and multiplication $(\times)$ to produce a sum (x_1+x_2) and product $(x_1 \times x_2)$, respectively, because X satisfies the five number field axioms [1, 2, 3]

1. X permits an element 1, namely the *multiplicative unit*, such that $1 \times x_A = x_A \times 1$, $\forall x_A \in X$;

2. X permits an element 0, namely the *additive unit*, such that $0 + x_A = x_A + 0$, $\forall x_A \in X$;

3. X is *closed* under addition and multiplication, which indicates that the respective sums $(x_A + x_B)$ and products $(x_A \times x_B)$ between elements $x_A, x_B \in X$ produce all possible elements of X;

4. X's addition and multiplication are *associative*, such that $(x_A + x_B) + x_C = x_A + (x_B + x_C)$ and $(x_A \times x_B) \times x_C = x_A \times (x_B \times x_C)$, respectively; and

5. the combination of X's addition and multiplication is *distributive*, such that $(x_A + x_B) \times x_C = x_A \times x_C + x_B \times x_C$ and $x_A \times (x_B + x_C) = x_A \times x_B + x_A \times x_C$.

A dominant problem of pure mathematics, in the context of number theory, is to establish a *universal number classification*, such as the identification of *all* sets that exist with numeric field axioms. To attack this *gigantic* classification problem, extensive and rigorous studies have been conducted over the course of history [1, 2, 4, 5, 6, 7]. A major result of these studies is the creation of real numbers [8, 9, 10], complex numbers [11, 12], and quaternion numbers [13, 14] with all possible numeric fields [2]. Moreover, it is known that these encoding frameworks bare *enormous* scientific application to quantifiable and computational implementations of disciplines such as physics, chemistry, biology, and more.

Fractal and chaotic patterns are *abundant* in the physical, chemical, and biological expressions of nature [15, 16, 17]. Fractal geometry—the language of chaos theory [18]—is a relatively new discipline of mathematics that was largely popularized by pioneer B. Mandelbrot [19, 20]. Chaos theory studies the behavior of dynamical systems that are *highly sensitive* to initial conditions [21, 15]. In a chaotic dynamical system, *miniscule* differences in initial conditions yield *widely diverging* outcomes, thereby generally rendering long-term predictions impossible [21, 15]. In addition to particle and astro physics, examples of chaos and fractals are also observed in lightning discharges [22, 23, 24, 25], weather patterns [26, 27, 28], aquatic ecosystems [29, 30], population biology [31], the biological allometric scaling laws [32, 33, 34, 35, 36], cancers and genetics [37, 38], viruses and pathogens [39, 40], the human brain [41, 42, 43], earthquakes [44, 45, 46], volcanoes [47, 48, 49], the global stock market [50, 51], and more. Certainly, iso-mathematics must play a fundamental role in classifying and demystifying such complex systems—but how?

As scientists and mathematicians, it is imperative to continue to investigate, scrutinize, challenge, develop, and test cutting-edge theories and ideas, such as the *triplex numbers* [52, 53, 54, 55], *Inopin's HR* [56], and *Santilli's iso-mathematics* [57, 58, 59, 60]. At the basis of this is the Scientific Method. Thus, in this preliminary paper, we hunt this "universal number classification beast" with an initial application of the Santillian iso-mathematics framework [2, 57, 58, 59, 60] to the triplex numbers [52], the Mandelbrot Set [19, 20], and the Inopin HR topology [56]. In Section 2, we prepare for the expedition by presenting a brief outline of Santilli's

framework [2, 57, 58, 59, 60], the triplex numbers [52], Mandelbrot's set [19, 20], and Inopin's HR topology [56]. Subsequently, in Section 3, we begin to apply Santilli's framework [2, 57, 58, 59, 60] to the triplex framework [52], Mandelbrot's set [19, 20], and Inopin's HR framework [56]. The paper concludes with Sections 4–5, where we briefly recapitulate the results and provide a thankful acknowledgment, respectively.

2 Alignment and background

Here, we prepare by aligning the reader with a background that highlights some aspects of Santilli's framework [2, 57, 58, 59, 60], the triplex numbers [52], Mandelbrot's set [19, 20], and Inopin's HR [56] that pertain to the application of Section 3.

2.1 Santilli's iso-mathematics framework

In [57, 58], Santilli reinspected the historical classification of sets verifying the numeric field axioms [1] and discovered that they equally authorize solutions for an arbitrary unit $\hat{r}$, generally outside the original fields [1], under the sole condition of being non-singular, and therefore invertible, such that $\hat{r} = \frac{1}{\hat{\kappa}}$, provided that the conventional associative number multiplication $x_A \times x_B$ is replaced with the associativity-preserving form [2, 57, 58]

$$x_A \hat{\times} x_B = x_A \times \hat{\kappa} \times x_B \tag{2}$$

under which $\hat{r}$ is indeed the right and left multiplicative unit. In [57, 58], Santilli then classified the new numbers depending on the main topological features of $\hat{r}$, such as:

1. $\hat{r}$ is single-valued and Hermitean for the case of *Santilli iso-numbers*;
2. $\hat{r}$ is single-valued and non-Hermitean for the case of *right Santilli geno-numbers*, characterized by $\hat{r}$, and *left Santilli geno-numbers* characterized by $\hat{r}^\dagger$; and
3. $\hat{r}$ is multi-valued and non-Hermitean for the case of *right Santilli hyper-numbers*, characterized by $\hat{r}$, and *left Santilli hyper-numbers* characterized by $\hat{r}^\dagger$.

Moreover, in [57, 58], Santilli identified the additional number series characterized by the anti-Hermitean image of the preceding generalized numbers via the iso-duality map that is denoted with the upper index $\hat{d} = \hat{r}^{\dagger}$ called *Santilli iso-dual iso-, geno-, and hyper-numbers.*

In general, Santilli [57, 58] successfully demonstrated that, in addition to conventional numbers, such axioms authorize the existence of four distinct iso-number classes [2]:

1. **Santilli iso-topic numbers ("iso-numbers")**

   - Iso-numbers exist because Santilli [57, 58] showed that the number field axioms [1] *do not* require that the multiplicative unit $\hat{r}$ is the number 1, so $\hat{r}$ can be any value provided that [2]:
     (a) the new *Santilli iso-unit* $\hat{r}$ is positive ($\hat{r} > 0$) to permit the inverse $\hat{r} = \frac{1}{\hat{\kappa}} > 0$,
     (b) the multiplication $x_A \times x_B$ is changed to the *Santilli iso-multiplication*

     $$x_A \hat{\times} x_B = x_A \times \hat{\kappa} \times x_B = x_A \times \frac{1}{\hat{r}} \times x_B, \tag{3}$$

     which is always associative for the *iso-topic liftings*

     $$\begin{aligned} x_A &= x_A \times \hat{r} \\ x_B &= x_B \times \hat{r}, \end{aligned} \tag{4}$$

     and
     (c) the additive unit and its sum are kept unchanged.
   - The number elements comprising X in eq. (1) are iso-topically "lifted" via the iso-topic lifting $X \to \hat{X}$ to become the *liftings* of the new iso-topic number set $\hat{X}$ $\hat{X}$ [2], which exist for the number field axioms [1] and are therefore numbers that are applicable to quantitative science [2]. Hence, the axiom of the multiplicative units is confirmed by the expression [2]

     $$1 \hat{\times} x_A = 1 \times \hat{\kappa} \times x_A = x_A \times \frac{1}{\hat{r}} \times 1 = x_A \hat{\times} 1 \tag{5}$$

     is valid $\forall x_A \in \hat{X}$.

- The original number field axioms [1] are preserved for $\hat{X}$ equipped with the iso-unit $\hat{r}$ for the iso-multiplication $x_A \hat{\times} x_B$ [2].
- The iso-multiplication led to additional refinement, identification, and usage [57, 58, 59] of the *iso-real numbers*, *iso-complex numbers*, and *iso-quaternion numbers* [2].
- Note the importance that if $x_A = 2$ and $x_B = 3$, then in general the iso-multiplication $2 \hat{\times} 3$ yields a product that is *different* than 6 [2].

2. **Santilli geno-topic numbers ("geno-numbers")**

- Geno-numbers exist because, in addition to iso-numbers, Santilli [57, 58] showed that the number field axioms [1] *do not* require that the iso-multiplication operates on both the right *and* left directions because the axioms are also tested when all the multiplications (and sums) are restricted to operate right $x_A \hat{\times}_> x_B$ *or* to operate left $x_A \hat{\times}_< x_B$ [2].
- When Santilli [57, 58] restricted all operations to act on the right or left, he was able to construct two different sets $\hat{X}_>$ and $\hat{X}_<$ with corresponding *Santilli geno-units* $\hat{r}_>$ and $\hat{r}_<$ for the compatible *Santilli geno-multiplication* operators $\hat{\times}_>$ and $\hat{\times}_<$, respectively [2].
- The geno-multiplication led to additional refinement, identification, and usage [57, 58, 59] of the *geno-real numbers*, *geno-complex numbers*, and *geno-quaternion numbers* [2].
- Note the importance that if $x_A = 2$ and $x_B = 3$, then in general the geno-multiplications $2 \hat{\times}_> 3$ and $2 \hat{\times}_< 3$ yield distinct products that both differ from 6 because $2 \hat{\times}_> 3 \neq 2 \hat{\times}_< 3$ and $\hat{r}_> \neq \hat{r}_<$ [2].

3. **Santilli hyper-topic numbers ("hyper-numbers" or "hyper-Santilli numbers")**

- Hyper-numbers (which are not to be confused with so-called hyper-mathematical structures that generally do not have units)

exist because, in addition to the geno-numbers, Santilli [57, 58] showed that a geno-multiplicative unit is not limited to a unique value because it can comprise a set of values for *Santilli hyper-multiplication*, such as the *Santilli hyper-units* $\hat{r}_{>} = \{1, 1, 2, 3, 5, ...\}$ or $\hat{r}_{>} = \{2, \frac{4}{5}, 7, ...\}$, if the set is ordered and defined as being applicable to right or left [2].

- The hyper-multiplication led to additional refinement, identification, and usage [57, 58, 59] of the *hyper-real numbers*, *hyper-complex numbers*, and *hyper-quaternion numbers* [2].
- Note the importance that, in general, the hyper-multiplications $2\ \hat{\times}_{>}\ 3$ and $2\ \hat{\times}_{<}\ 3$ yield two distinct result sets that both differ from 6 [2].

4. **Santilli iso-dual numbers**

- *Iso-dual numbers* exist because, in addition to iso-numbers, geno-numbers, and hyper-numbers, Santilli [57, 58] showed that the multiplicative unit $\hat{r}$ can be any (positive or negative) value except for zero (i.e. $-\hat{r}$) [2].
- The "iso-dual" term identifies a duality between positive and negative units in accordance to the original number field axioms [2].
- The *iso-dual multiplication* led to additional refinement, identification, and usage [57, 58, 59] *iso-dual iso-numbers*, *iso-dual geno-numbers*, and *iso-dual hyper-numbers* [2].

So in total, Santilli's axiomatic iso-mathematics framework [57, 58, 59, 60] reveals *eleven* new data structures [2]:

- iso-numbers,
- geno-numbers (right and left),
- hyper-numbers (right and left),
- conventional iso-dual numbers,

- iso-dual iso-numbers,
- iso-dual geno-numbers (right and left), and
- iso-dual hyper-numbers (right and left);

each data structure is applicable to the real [8, 9, 10], complex [11, 12], and quaternion numbers [13, 14], where each application bares an infinite number of possible units [2].

Subsequently, in a series of physical implementations, Santilli [57, 58, 59, 60] then utilized the generalized iso-numbers, geno-numbers, and hyper-numbers to characterize the increasing complexities of matter with regard to non-linearity, non-Hamiltonian features, irreversibility, multi-valuedness, etc., while the iso-dual images were used by Santilli to characterize anti-matter under the corresponding increase of complexity. For an in-depth explanation of this framework, we recommend a technical study of the original publications [57, 58, 59, 60].

2.2 Complex numbers, Euclidean complex space, Mandelbrot set, and Inopin 1-sphere HR topology

Chronologically, the establishment of the Inopin HR [56] came before the triplex numbers [52]. Inopin's HR was initially introduced in the analytic quark confinement and baryon-antibaryon duality proof of [56]. The HR is a powerful tool because it is topological sphere with an "amplitude-radius" (or "amplitude-modulus") that serves as an *iso-metric* to construct Inopin's HR topology, which may be utilized to attack a wide range of mathematical and physical problems [52, 56]. In this section, we recall the synchronized complex number and 2D coordinate-vector system for Inopin's 1-sphere HR topology, which are fundamental prerequisites of the synchronized triplex number and 3D coordinate-vector system for Inopin's 2-sphere HR topology [52, 56] in the upcoming Section 2.3.

First, starting from eq. (7) in [52], we engage the complex number, 2D polar coordinate-vector, and 2D Cartesian coordinate-vector synchronization form

$$x = \vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}} = (\vec{x}) = (|\vec{x}|, \langle \vec{x} \rangle)_P = (\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C, \quad \forall \vec{x} \in X, \tag{6}$$

where the complex number $\vec{x}$ is a *dual 2D Cartesian-polar coordinate-vector state* in the *dual 2D Cartesian-polar coordinate-vector state space* and Euclidean complex space X, such that

$$X = \mathbb{C} \tag{7}$$

is also the set of all complex numbers (a field): $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C$ is a *2D Cartesian coordinate-vector state* in the *2D Cartesian coordinate-vector state space* X_C so $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$, while $(|\vec{x}|, \langle\vec{x}\rangle)_P$ is a *2D polar coordinate-vector state* in the *2D polar coordinate-vector state space* X_P so $(|\vec{x}|, \langle\vec{x}\rangle)_P \in X_P$, where X_C and X_P are iso-morphic, dual, synchronized, and interlocking in X [52, 56]. To prove this for eq. (6), the transformations between X_C and X_P, $\forall(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$ and $\forall(|\vec{x}|, \langle\vec{x}\rangle)_P \in X_P$, are identified by the complex number-coordinate-vector constraints of eqs. (2–6) in [52], which define:

1. the $|\vec{x}| \in [0, \infty)$ *amplitude* ("radius" or "modulus" or "Euclidean distance") polar component

$$|\vec{x}| = \sqrt{\vec{x}_{\mathbb{R}}^{\,2} + \vec{x}_{\mathbb{I}}^{\,2}} \; ; \tag{8}$$

2. the $\langle\vec{x}\rangle \in [0, 2\pi]_{\mathbb{RI}}$ *phase* ("azimuth" or "$\mathbb{RI}$-direction") polar component (for the $\mathbb{RI}$-plane)

$$\langle\vec{x}\rangle = \arctan\left(\frac{\vec{x}_{\mathbb{I}}}{\vec{x}_{\mathbb{R}}}\right); \tag{9}$$

3. the $\vec{x}_{\mathbb{R}} \in (-\infty_{\mathbb{R}}, \infty_{\mathbb{R}})$ *real* ("x-axis") Cartesian component

$$\vec{x}_{\mathbb{R}} = \frac{\vec{x}_{\mathbb{I}}}{\tan\langle\vec{x}\rangle} = |\vec{x}|\cos\langle\vec{x}\rangle \; ; \tag{10}$$

and

4. the $\vec{x}_{\mathbb{I}} \in (-\infty_{\mathbb{I}}, \infty_{\mathbb{I}})$ *imaginary* ("y-axis") Cartesian component

$$\vec{x}_{\mathbb{I}} = i\vec{x}_{\mathbb{R}}\tan\langle\vec{a}\rangle = i|\vec{x}|\sin\langle\vec{x}\rangle. \tag{11}$$

Hence, for X, the built-in iso-morphic duality between X_C and X_P is defined as

$$\delta_{2D} : X_C \rightarrow X_P \tag{12}$$

from the transformation of eqs. (8–9) and

$$\delta_{2D}^{-1} : X_P \rightarrow X_C \tag{13}$$

from the transformation of eqs. (10–11). Therefore, X clearly satisfies the five number field axioms of [1]. For a depiction of eqs. (6–13) see Figure 1, where observe that $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ are treated as vectors (with axis-dependent magnitude and direction) so the vector sum is $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ with amplitude $|\vec{x}|$ and direction $\langle \vec{x} \rangle$.

Second, given that X is the set of complex numbers, we note that X can be populated with the Mandelbrot Set by systematically iterating Mandelbrot's complex quadratic polynomial [19, 20]

$$\vec{x}_{n+1} = \vec{x}_n^{\,2} + \vec{c} = \vec{x}_{n+1_{\mathbb{R}}} + \vec{x}_{n+1_{\mathbb{I}}} = (\vec{x}_{n_{\mathbb{R}}} + \vec{x}_{n_{\mathbb{I}}})^2 + (\vec{c}_{\mathbb{R}} + \vec{c}_{\mathbb{I}}), \tag{14}$$

where $\vec{x}_n, \vec{x}_{n+1}, \vec{c} \in X$ are complex numbers that satisfy eqs. (6–13)—see Figure 2.

Third, following [52, 56], we equip X with Inopin's 1-sphere HR, namely $T^1 \subset X$, by iso-metrically embedding T^1 in X of eqs. (10–11) of [56] to construct Inopin's 1-sphere HR topology; T^1 is a topological circle that is centered on the origin

$$O \in X = 0 + 0i \tag{15}$$

with the *positive-definite* amplitude-radius r and the corresponding curvature $\kappa = \frac{1}{r}$. Hence, T^1 is employed to topologically encode the 1-sphere version of Inopin's "non-linear time dimension" and "temporal distance scale", where eq. (15) of [56] defines it as

$$T^1 = \{\vec{x} \in X : |\vec{x}| = r\} \tag{16}$$

for Inopin's 1-sphere HR topology, which is the multiplicative group of all non-zero complex numbers with the amplitude-radius r.

Now T^1 was equipped with topological deformation order parameter fields of fractional statistics for quasi-particles [52, 56]. The *inside* of T^1

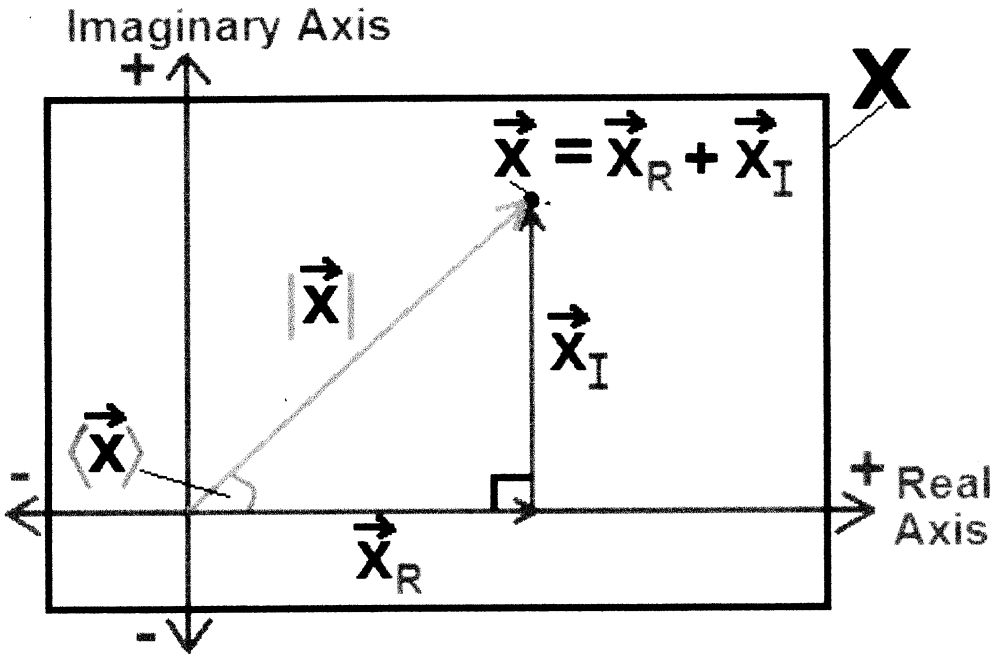

Fig. 1: Complex components for the dual 2D Cartesian-polar coordinate-vector state $\vec{x}$ in the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X, such that $\vec{x} \in X$, where $\vec{x}$ is simultaneously treated as a complex number, 2D polar coordinate-vector, and 2D Cartesian coordinate-vector. Specifically, $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C$ is a 2D Cartesian coordinate-vector state in the 2D Cartesian coordinate-vector state space X_C so $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$, while $(|\vec{x}|, \langle\vec{x}\rangle)_P$ is a 2D polar coordinate-vector state in the 2D polar coordinate-vector state space X_P so $(|\vec{x}|, \langle\vec{x}\rangle)_P \in X_P$, where X_C and X_P are iso-morphic, dual, synchronized, and interlocking in X. Note that $\vec{x}_{\mathbb{R}}$ and $\vec{x}_{\mathbb{I}}$ are treated as vectors (with axis-dependent magnitude and direction) so the vector sum is $\vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}}$ with amplitude $|\vec{x}|$ and direction $\langle\vec{x}\rangle$.

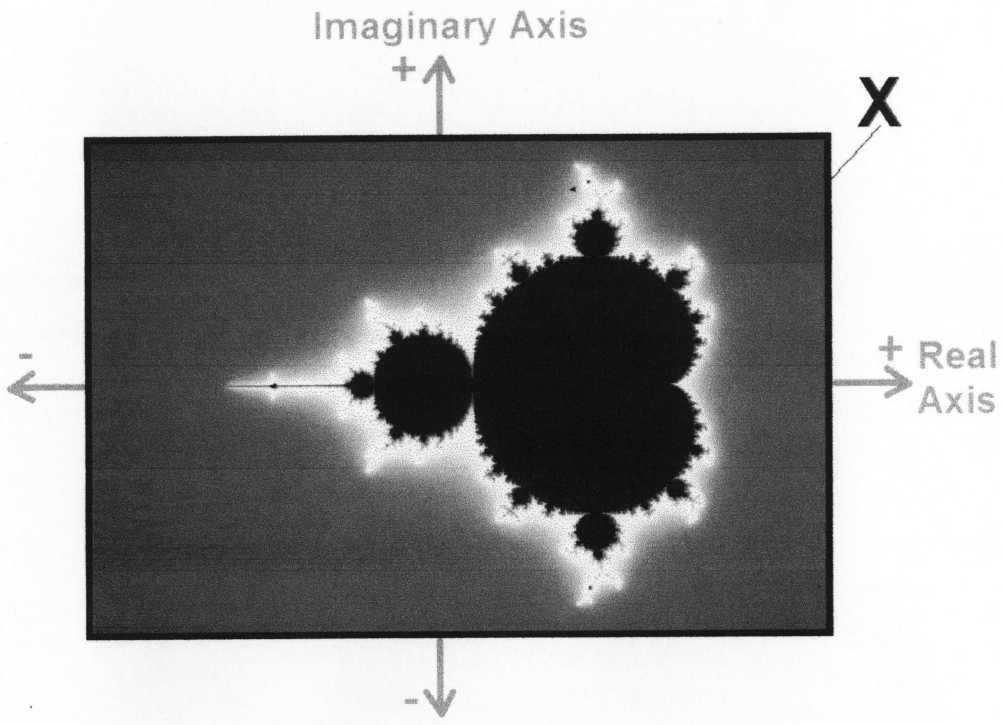

Fig. 2: The Mandelbrot Set [19, 20], which is often considered to be the most famous fractal, can populate the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X.

corresponds to an *interior* dynamical system of *superluminal quasi-particle spatial excitations*, namely the *2-brane micro sub-space zone* (or "short spatial distance scale") $X_- \subset X$ of eq. (16) in [56], while the *outside* of T^1 corresponds to an *exterior* dynamical system of *luminal or sub-luminal quasi-particle spatial excitations*, namely the *2-brane macro sub-space zone* (or "long spatial distance scale") $X_+ \subset X$ of eq. (17) in [56]—T^1 itself is populated with *luminal quasi-particle temporal excitations*, which is simultaneously dual to both X_- and X_+ as in eqs. (20–21) in [56]. Here, $X_-, T^1, X_+ \subset X$ are disjoint and comprise the complete X, such that $X = X_- \cup T^1 \cup X_+$ [56]—see Figure 3. For this, each complex location $\vec{x} \in X$ was equipped with one or more complex order parameter field states in the generic form $\vec{\psi}(\vec{x})$ within a complex order parameter field state space $\Phi(\vec{x})$ for topological deformations, such that $\vec{\psi}(\vec{x}) \in \Phi(\vec{x})$—see eq. (20) in [52].

For the quark confinement proof of [56], the three distinct quark-antiquark pairs for a baryon-antibaryon pair are confined to T^1 in the upgraded Gribov vacuum on a six-coloring kagome lattice antiferromagnet with correlated order parameters. The transforming wavefunction states of T^1 are directly related to the states of X_- and X_+, which are 2-branes within 3-branes, so their 3D state space is directly inferred from the 2D state space of T^1. But at the time of writing [56], the triplex numbers with triplex multiplication did not yet exist, therefore it was not possible to fully encode the 3D state space of the 3-branes—this genuine need to employ 3D numbers to encode 3D states (with a pertinent 3D number multiplication) fueled the motivation for the triplex implementation of [52], which served as a topological upgrade to [56].

2.3 Triplex numbers, Euclidean triplex space, Mandelbulb set, and Inopin 2-sphere HR topology

The triplex work of [52] was inspired by the emerging "3D hyper-complex number" framework of pioneers D. White and P. Nylander [53, 54, 55]. White and Nylander originally developed their 3D hyper-complex number system in order to generate magnificent 3D fractals with computer graphics [53, 54, 55]. However, the White-Nylander 3D hyper-complex approach [53, 54] was still considered incomplete primarily because their 3D spherical

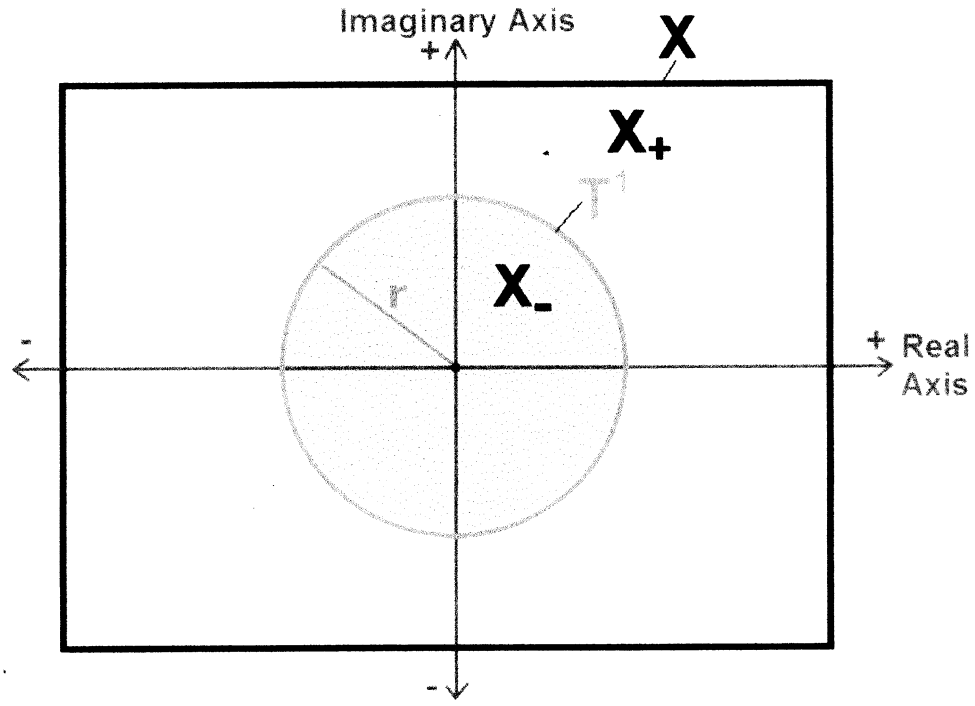

Fig. 3: Inopin's 1-sphere HR topology for the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X, where the topological 1-sphere HR $T^1 \subset X$ is simultaneously dual to two spatial 2-branes [52, 56]: the micro sub-space zone $X_- \subset X$ and the macro sub-space zone $X_+ \subset X$ for interior and exterior dynamical systems, respectively.

form (extended polar form) was not unique while the system did not form a well-behaved algebra [55, 52]. Fortunately, after a thorough and rigorous investigation, this great "White-Nylander mythical beast" of [53, 54, 55] was destroyed via the creation of a well-behaved triplex algebra equipped with a unique 3D spherical form and triplex multiplication—observe the work of [52] and the upcoming eqs. (17–28). Thus, the triplex structure Y was introduced to supersede the complex structure X by upgrading it with a third axis for 3D representation, such that $X \subset Y$, to establish Inopin's 2-sphere HR topology [52].

First, eq. (35) in [52] defines the triplex number, 3D spherical coordinate-vector, and 3D Cartesian coordinate-vector synchronization form

$$y = \vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_{Z} = (\vec{y}) = (|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S = (\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C, \quad \forall \vec{y} \in Y, \quad (17)$$

where the triplex number $\vec{y}$ is a *dual 3D Cartesian-spherical coordinate-vector state* in the *dual 3D Cartesian-spherical coordinate-vector state space* and Euclidean triplex space Y, such that

$$Y \equiv \mathbb{T} \quad (18)$$

is also defined as the set of all triplex numbers (a field): $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C$ is a *3D Cartesian coordinate-vector state* in the *3D Cartesian coordinate-vector state space* Y_C so $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C \in Y_C$, while $(|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S$ is a *3D spherical coordinate-vector state* in the *3D spherical coordinate-vector state space* Y_S so $(|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S \in Y_S$, where Y_C and Y_S are iso-morphic, dual, synchronized, and interlocking in Y [52]. To prove this for eq. (17), the transformations between Y_C and Y_S, $\forall (\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_{Z})_C \in Y_C$ and $\forall (|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S \in Y_S$, are identified by the triplex number and 3D coordinate-vector constraints of eqs. (29–34) in [52], which define:

1. the $|\vec{y}| \in [0, \infty)$ *amplitude* ("radius" or "modulus" or "Euclidean distance") spherical component

$$|\vec{y}| = \sqrt{\vec{y}_{\mathbb{R}}^{\,2} + \vec{y}_{\mathbb{I}}^{\,2} + \vec{y}_{Z}^{\,2}} \; ; \quad (19)$$

2. the $\langle \vec{y} \rangle \in [0, 2\pi]_{\mathbb{RI}}$ *phase* ("azimuth" or "$\mathbb{RI}$-direction") spherical component (for the $\mathbb{RI}$-plane)

$$\langle \vec{y} \rangle = \arctan\left(\frac{\vec{y}_{\mathbb{I}}}{\vec{y}_{\mathbb{R}}}\right) ; \quad (20)$$

3. the $[\vec{y}] \in [0, 2\pi]_{\mathbb{R}Z}$ *inclination* ("zenith" or "$\mathbb{R}Z$-direction") spherical component (for the $\mathbb{R}Z$-plane)

$$[\vec{y}] = \arctan\left(\frac{\vec{y}_Z}{\vec{y}_{\mathbb{R}}}\right); \tag{21}$$

4. the $\vec{y}_{\mathbb{R}} \in (-\infty_{\mathbb{R}}, \infty_{\mathbb{R}})$ *real* ("x-axis") Cartesian component

$$\vec{y}_{\mathbb{R}} = \frac{\vec{y}_{\mathbb{I}}}{\tan\langle\vec{y}\rangle} = |\vec{y}|_{\mathbb{R}\mathbb{I}} \cos\langle\vec{y}\rangle \; ; \tag{22}$$

5. the $\vec{y}_{\mathbb{I}} \in (-\infty_{\mathbb{I}}, \infty_{\mathbb{I}})$ *imaginary* ("y-axis") Cartesian component

$$\vec{y}_{\mathbb{I}} = i\vec{y}_{\mathbb{R}} \tan\langle\vec{y}\rangle = i|\vec{y}|_{\mathbb{R}\mathbb{I}} \sin\langle\vec{y}\rangle, \tag{23}$$

where

$$|\vec{y}|_{\mathbb{R}\mathbb{I}} = \sqrt{\vec{y}_{\mathbb{R}}^{\,2} + \vec{y}_{\mathbb{I}}^{\,2}} \tag{24}$$

is the 2D limited amplitude radius for the $\mathbb{R}\mathbb{I}$ plane; and

6. the $\vec{y}_Z \in (-\infty_Z, \infty_Z)$ *projected* ("z-axis") Cartesian component

$$\vec{y}_Z = j\vec{y}_{\mathbb{R}} \tan[\vec{y}] = j|\vec{y}|_{\mathbb{R}Z} \sin[\vec{y}], \tag{25}$$

where

$$|\vec{y}|_{\mathbb{R}Z} = \sqrt{\vec{y}_{\mathbb{R}}^{\,2} + \vec{y}_Z^{\,2}} \tag{26}$$

is the 2D limited amplitude radius for the $\mathbb{R}Z$ plane.

Hence, for Y, the built-in iso-morphic duality between Y_C and Y_S is defined as

$$\delta_{3D} : Y_C \to Y_S \tag{27}$$

from the transformation of eqs. (19–21) and

$$\delta_{3D}^{-1} : Y_S \to Y_C \tag{28}$$

from the transformation of eqs. (22–26). For a depiction of eqs. (17–28) see Figures 4–5, where observe that $\vec{y}_{\mathbb{R}}$, $\vec{y}_{\mathbb{I}}$, and $\vec{y}_Z$ are treated as vectors

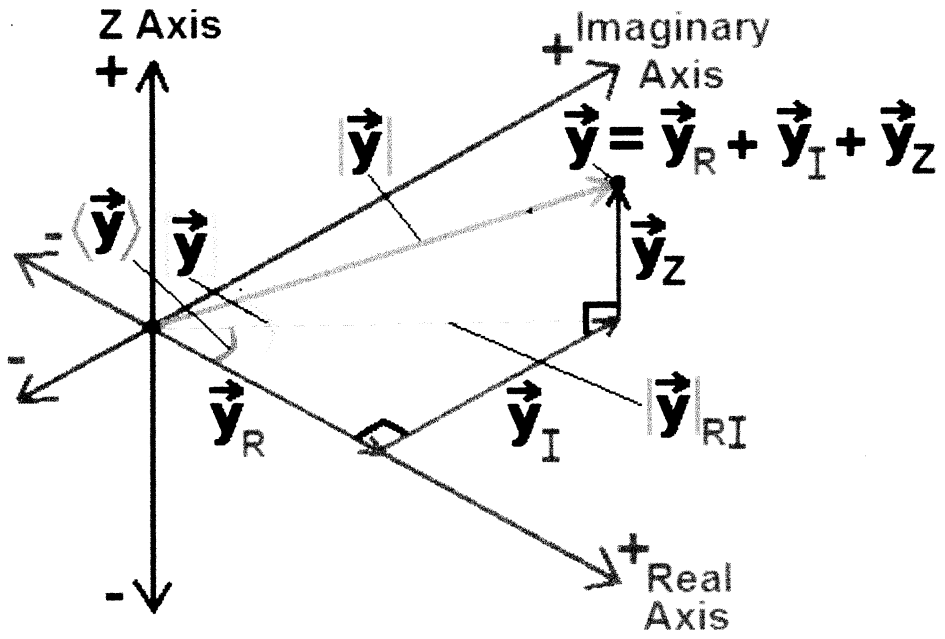

Fig. 4: Triplex components for the dual 3D Cartesian-spherical coordinate-vector state $\vec{y}$ in the dual 3D Cartesian-polar coordinate-vector state space (and Euclidean triplex space) Y, such that $\vec{y} \in Y$, where $\vec{y}$ is simultaneously treated as a triplex number, 3D spherical coordinate-vector, and 3D Cartesian coordinate-vector. Specifically, $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C$ is a 3D Cartesian coordinate-vector state in the 3D Cartesian coordinate-vector state space Y_C so $(\vec{y}_{\mathbb{R}}, \vec{y}_{\mathbb{I}}, \vec{y}_Z)_C \in Y_C$, while $(|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S$ is a 3D spherical coordinate-vector state in the 3D spherical coordinate-vector state space Y_S so $(|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S \in Y_S$, where Y_C and Y_S are iso-morphic, dual, synchronized, and interlocking in Y. Note that $\vec{y}_{\mathbb{R}}$, $\vec{y}_{\mathbb{I}}$, and $\vec{y}_Z$ are treated as vectors (with axis- dependent magnitude and direction) so the vector sum is $\vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_Z$ with amplitude $|\vec{y}|$ and two directions $\langle \vec{y} \rangle$ and $[\vec{y}]$.

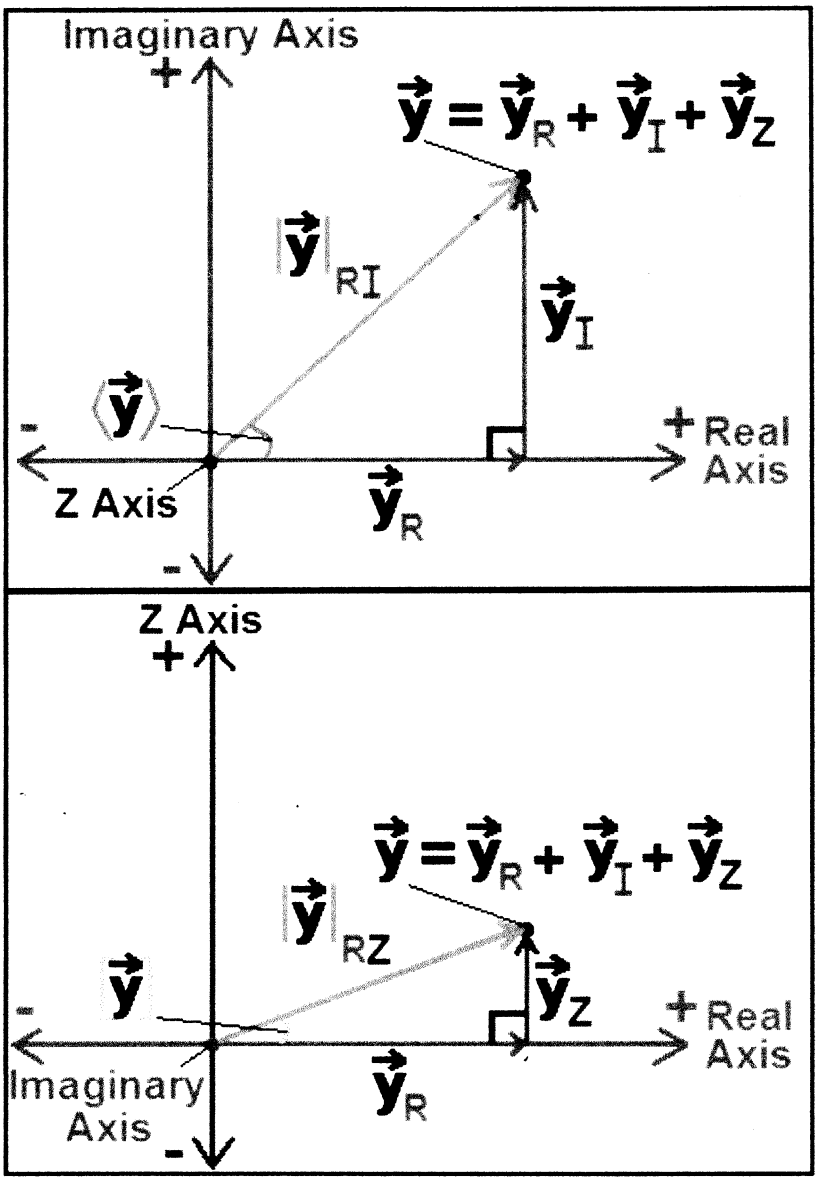

Fig. 5: Aligned perspectives of $\vec{y} \in Y$ from the $\mathbb{R}\mathbb{I}$-plane (top) and the $\mathbb{R}Z$-plane (bottom).

(with axis-dependent magnitude and direction) so the vector sum is $\vec{y} = \vec{y}_{\mathbb{R}} + \vec{y}_{\mathbb{I}} + \vec{y}_Z$ with amplitude $|\vec{y}|$ and two directions $\langle\vec{y}\rangle$ and $[\vec{y}]$.

Second, using the notation of eq. (17), the triplex multiplication between the two distinct triplex numbers $\vec{y}_A, \vec{y}_B \in Y$ to yield the product $\vec{y}_C \in Y$ is defined in eqs. (70–71) of [52] as

$$\vec{y}_C = \vec{y}_A \times \vec{y}_B \iff \begin{array}{rcl} |\vec{y}_C| & = & |\vec{y}_A| \times |\vec{y}_B| \\ \langle\vec{y}_C\rangle & = & \langle\vec{y}_A\rangle + \langle\vec{y}_B\rangle \\ [\vec{y}_C] & = & [\vec{y}_A] + [\vec{y}_B] \end{array}, \quad \forall \vec{y}_A, \vec{y}_B, \vec{y}_C \in Y. \tag{29}$$

In other words, the amplitude-radius values are multiplied while the corresponding directional values are added to yield the triplex spherical form of the product in eq. (29). Hence, given that division is defined as the inverse, opposite, and reverse operation of multiplication, then we engage eq. (29) to incorporate this fundamental relation and thereby define the *triplex division* as

$$\vec{y}_C \equiv \vec{y}_A \,/\, \vec{y}_B \iff \begin{array}{rcl} |\vec{y}_C| & \equiv & |\vec{y}_A| \,/\, |\vec{y}_B| \\ \langle\vec{y}_C\rangle & \equiv & \langle\vec{y}_A\rangle - \langle\vec{y}_B\rangle \\ [\vec{y}_C] & \equiv & [\vec{y}_A] - [\vec{y}_B] \end{array}, \quad \forall \vec{y}_A, \vec{y}_B, \vec{y}_C \in Y, \tag{30}$$

where $\vec{y}_A$ is the dividend, $\vec{y}_B$ is the divisor, and $\vec{y}_C$ is the quotient because eq. (30) comprises operations that are the inverse, opposite, and reverse of eq. (29). In other words, the amplitude-radius dividend is divided by the amplitude-radius divisor while the corresponding directional values are subtracted to yield the triplex spherical form of the quotient in eq. (30). Therefore, Y clearly satisfies the five number field axioms of [1].

Third, we summarize the pertinent aspects of P. Rowlands' dual 3D vector space [61, 62, 63], which is a quaternion-based, nilpotent implementation of our Y. According to [61, 62, 63], a triplex structure based on three orthogonal axes—i.e. the $\mathbb{R}$-axis, the $\mathbb{I}$-axis, and the Z-axis of Y—can be constructed with a dual, interlocking 3D vector system that generates a double 3D Clifford algebra to encode spatial states of dynamical systems [61, 62, 63]. An overview of this process is as follows [61, 62, 63]:

1. At the start, two vector spaces are defined in the Clifford algebra formulation, where the vector spaces are *distinct but identical* in every

respect [61, 62, 63]. Both vector spaces have, in addition to one scalar unit, three vector units to encode lengths, three bivector units to encode areas, and one pseudo-scalar unit to encode volume [61, 62, 63]. Vectors in this algebra have a complete product and well-behaved multiplication—see eq. (3) of [61].

2. Once the dual vector spaces are assembled, a commutative combination of the two spaces (allowing for + and 0 signs) generates an algebra structured on 64 units so the Clifford algebra vectors can also be written as complexified quaternions [61, 62, 63]. Therefore, the complete double algebra is encoded as either a complexified double quaternion algebra or as a product of a vector and a quaternion algebra [61, 62, 63]—this yields a total of three unit sets of order 64 that are iso-morphic to, for example, the Dirac algebra [61, 62, 63].

3. Now each unit set can be created by five generators [61, 62, 63]. There are many options for the generators, but they all have the same structure in which the symmetry of one 3D unit set is preserved, while the other is broken [61, 62, 63]. If six generators are used then both symmetries are preserved, but not in the minimum case of using five generators [61, 62, 63]: there are *three* instances of the five generators that preserve one symmetry but not the other [61, 62, 63].

4. These three resulting instances permit the creation of a triplex representation whose symmetry is not preserved as the basis of the three triplex axes [61, 62, 63]. The character of each axis is then determined by the type of quantity in which it is multiplied by [61, 62, 63].

5. Next, the middle three terms of a given five generator are used to encode the Z-axis as a vector, and the three vector units in each case are combined to be a single vector unit [61, 62, 63]. This yields a triplex structure where the axes are encoded as quaternions [61, 62, 63].

6. Interestingly, Rowlands' triplex structure can be thought of as being an endless fractal sequence as one moves up the scale [63]. The first triplex structure, which is originally unbroken, becomes broken

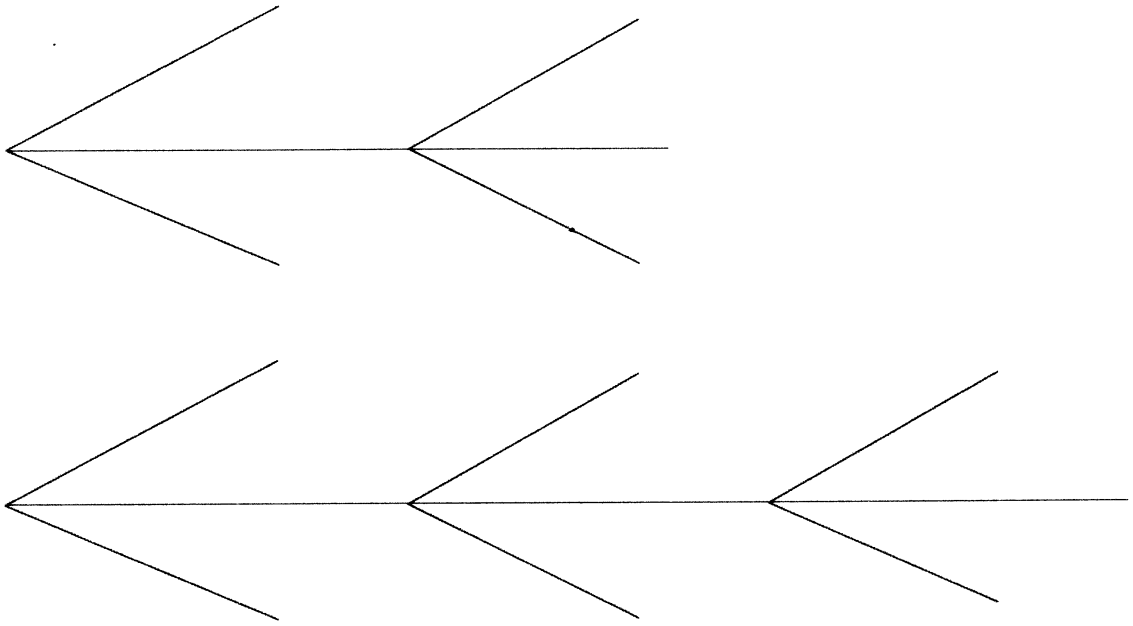

Fig. 6: Rowlands' triplex structure can be thought of as being an endless fractal sequence as one moves up the scale [63]. The first triplex structure, which is originally unbroken, becomes broken by having a second triplex structure attached (top). Consequently, the second triplex structure is now unbroken, but becomes broken by attaching another triplex structure (bottom), and so on indefinitely.

> by having a second triplex structure attached [63]. Consequently, the second triplex structure is now unbroken, but becomes broken by attaching another triplex structure, and so on indefinitely [63]. See Figure 6 for a depiction of this process [63].

Fourth, given that Y is the set of triplex numbers equipped with a well-behaved triplex algebra and unique triplex spherical form [52] that is similar to Rowlands' approach [61, 62, 63] and extends the White-Nylander approach [53, 54, 55], we note that Y can be populated with the Mandelbulb Set—a fundamental 3D fractal—by systematically iterating Mandelbrot's

triplex quadratic polynomial

$$\vec{y}_{n+1} = \vec{y}_n{}^2 + \vec{c} = \vec{y}_{n+1_{\mathbb{R}}} + \vec{y}_{n+1_{\mathbb{I}}} + \vec{y}_{n+1_Z} = (\vec{y}_{n_{\mathbb{R}}} + \vec{y}_{n_{\mathbb{I}}} + \vec{y}_{n_Z})^2 + (\vec{c}_{\mathbb{R}} + \vec{c}_{\mathbb{I}} + \vec{c}_Z), \quad (31)$$

where $\vec{y}_n, \vec{y}_{n+1}, \vec{c} \in \mathbb{Y}$ are triplex numbers that satisfy eqs. (17–30).

Fifth, following [52], we equip Y with Inopin's 2-sphere HR, namely $T^2 \subset Y$, by iso-metrically embedding T^2 in Y of eqs. (40–41) of [52] to construct Inopin's 2-sphere HR topology; T^1 is a topological 2-sphere that is centered on the origin

$$O \in Y = 0 + 0i + 0j \quad (32)$$

with the amplitude-radius r and the corresponding curvature $\kappa = \frac{1}{r}$ (the same as T^1). Hence, T^2 is employed to topologically encode the 2-sphere version of Inopin's "non-linear time dimension" and "temporal distance scale", where eq. (40) of [52] defines it as

$$T^2 = \{\vec{y} \in Y : |\vec{y}| = r\} \quad (33)$$

for Inopin's 2-sphere HR topology, which is the multiplicative group of all non-zero triplex numbers with the amplitude-radius r [52]—$T^1 \subset X$ is generalized to $T^2 \subset Y$ [52]. Hence, given that $X \subset Y$ and $T^1 \subset X \subset Y$, then

$$T^1 = T^2 \cap X, \quad (34)$$

so T^1 is the great (topological) circle of T^2 [52, 56], such that $T^1 \subset T^2$, where both T^1 and T^2 share the same positive-definite amplitude-radius r and curvature $\kappa = \frac{1}{r}$. So given $X \to Y$ and $X \subset Y$, T^2 delineates the dual interconnected spatial 3-branes $Y_- \subset Y$ and $Y_+ \subset Y$ in eq. (41) of [52], which supercede the spatial 2-branes $X_- \subset Y_- \subset Y$ and $X_+ \subset Y_+ \subset Y$, where T^2 is simultaneously dual to the *3-brane micro sub-space zone* Y_- and the *3-brane macro sub-space zone* Y_+ in eq. (44) of [52] for $Y = Y_- \cup T^2 \cup Y_+$, recalling that $Y_-, T^2, Y_+ \subset Y$ are disjoint [52]—see Figure 7. For this, we observe the analogy: T^1 *is to* $X_-, X_+ \subset X$ *just as* T^2 *is to* $Y_-, Y_+ \subset Y$, *respectively.* Here, we recall from [52] that each triplex location $\vec{y} \in Y$ was equipped with one or more triplex order parameter field states in the generic form $\vec{\psi}(\vec{y})$ within a triplex order parameter field state space $\Phi(\vec{y})$ for topological deformations, such that $\vec{\psi}(\vec{y}) \in \Phi(\vec{y})$—see eq. (50) in [52].

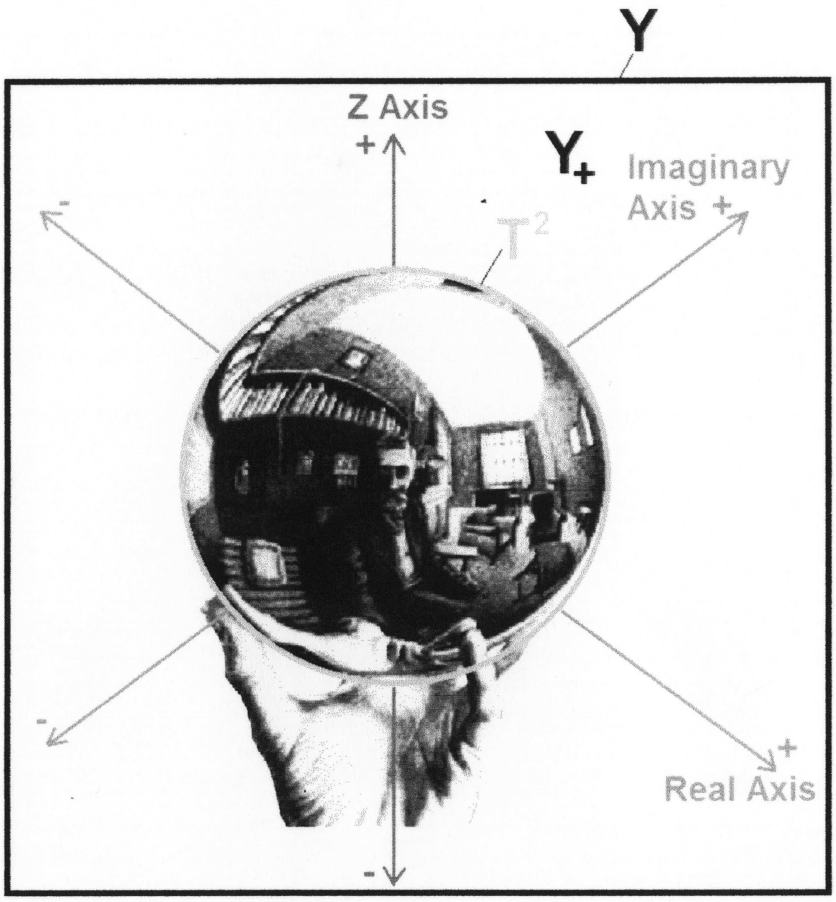

Fig. 7: Inopin's 2-sphere HR topology in the dual 3D Cartesian-spherical coordinate-vector state space (and Euclidean triplex space) Y, where the topological 2-sphere HR $T^2 \subset Y$ is simultaneously dual to two spatial 3-branes [52, 56]: the micro sub-space zone $Y_- \subset Y$ and the macro sub-space zone $Y_+ \subset Y$ for interior and exterior dynamical systems, respectively. Here, T^2 is depicted as M. C. Escher's famous reflecting sphere [64].

Therefore, given that Santilli applied his iso-mathematics [57, 58] to the real [8, 9, 10], complex [11, 12], and quaternion numbers [13, 14], it seems important that his breakthroughs [2, 57, 58, 59, 60] should also be applied to the triplex numbers [52], where the amplitude-radius r of both T^1 and T^2 serves as an iso-metric [52, 56].

3 Application

In this section, we begin to apply Santilli's iso-mathematics framework [2, 57, 58, 59, 60] to the triplex numbers [52] and Inopin's 2-sphere HR topology [52, 56]. In Section 3.1, our first attack focuses on iso-topically lifting the triplex number space Y—*excluding* T^2—and subsequently gives a preliminary assessment on the feasibility of iso-morphing Y via geno-topic, hyper-topic, and iso-dual-topic liftings. Next, we advance to our second engagement in Section 3.2, where we assemble a preliminary explanation on how one can upgrade the Y—*including* T^2—of Inopin's 2-sphere HR topology [52, 56] with Santilli's iso-numbers [2, 57, 58, 59, 60]. In both phases, we report the initial results.

3.1 Iso-triplex numbers, Euclidean iso-triplex space, and iso-fractal Mandelbulb Set

Here, in the first phase of engagement, the initial and primary objective is to iso-topically lift Y in accordance to Santilli's iso-numbers [2, 57, 58, 59, 60], while the secondary objective is to assess the possibility of iso-morphing Y via geno-topic, hyper-topic, and iso-dual-topic liftings.

First, for Santilli's iso-topic numbers [2, 57, 58, 59, 60], we select some $\hat{r} > 0$ with corresponding inverse $\hat{\kappa} = \frac{1}{\hat{r}}$, such that $\hat{\kappa} > 0$. Next, following [2, 57, 58, 59, 60], Y's triplex numbers $\vec{y}_A, \vec{y}_B \in Y$ are redefined in the form

$$\begin{array}{rcl} \vec{y}_A & \equiv & \vec{y}_A \times \hat{r} \\ \vec{y}_B & \equiv & \vec{y}_B \times \hat{r} \end{array}, \quad \forall \vec{y}_A, \vec{y}_B \in Y \rightarrow \forall \vec{y}_A, \vec{y}_B \in \hat{Y}, \tag{35}$$

where the triplex number set and Euclidean triplex space Y of eq. (18) is iso-topically lifted via $Y \rightarrow \hat{Y}$ to become $\hat{Y}$—defined as both the *iso-triplex number set* and the *Euclidean iso-triplex space*—such that the triplex multiplication between $\vec{y}_A, \vec{y}_B \in Y$ of eq. (29) is upgraded to define the *iso-*

triplex multiplication as

$$\vec{y}_A \hat{\times} \vec{y}_B \equiv \vec{y}_A \times \hat{\kappa} \times \vec{y}_B = \vec{y}_A \times \frac{1}{\hat{r}} \times \vec{y}_B \tag{36}$$

for $\hat{Y}$, which is always associative, such that the additive unit and its sum remain unmodified. Hence, from [2, 57, 58, 59, 60], the axiom of multiplicative units is confirmed by the expressions

$$\begin{aligned} 1 \hat{\times} \vec{y}_A &\equiv 1 \times \hat{\kappa} \times \vec{y}_A \equiv \vec{y}_A \times \tfrac{1}{\hat{r}} \times 1 \equiv \vec{y}_A \hat{\times} 1 \\ 1 \hat{\times} \vec{y}_B &\equiv 1 \times \hat{\kappa} \times \vec{y}_B \equiv \vec{y}_B \times \tfrac{1}{\hat{r}} \times 1 \equiv \vec{y}_B \hat{\times} 1, \end{aligned} \tag{37}$$

which are valid $\forall \vec{y}_A, \vec{y}_B \in Y \rightarrow \forall \vec{y}_A, \vec{y}_B \in \hat{Y}$. Therefore, the operation $\vec{y}_A \hat{\times} \vec{y}_B$ of eq. (36) is known as the triplex version of Santilli iso-multiplication, so we've employed the work of [57, 58, 59, 60] to identify the Santilli iso-triplex number set in accordance to [2]. Therefore, the results of eqs. (36–37) indicate that the iso-topic lifting and its inverse are

$$\begin{aligned} f(\hat{r}): \quad & Y \rightarrow \hat{Y} \\ f^{-1}(\hat{r}): \quad & \hat{Y} \rightarrow Y, \end{aligned} \tag{38}$$

respectively, which define an iso-morphism (or bijective homo-morphism), where $\hat{Y}$ is the Euclidean iso-triplex space. At this point, we've successfully applied Santilli's iso-numbers [57, 58, 59, 60] to the triplex numbers [52] so we identify:

- **Lemma 1.1**: A Euclidean triplex space Y (equipped with the multiplication operator $\times$) can be iso-topically lifted via the transition $f(\hat{r}) : Y \rightarrow \hat{Y}$ (with iso-unit $\hat{r}$) to become a Euclidean iso-triplex space $\hat{Y}$ (equipped with the iso-multiplication operator $\hat{\times}$), where Y and $\hat{Y}$ are locally iso-morphic.

- **Lemma 1.2**: A Euclidean triplex space Y and its corresponding (iso-topically lifted) Euclidean iso-triplex space $\hat{Y}$ are locally iso-morphic because:

  1. Y and $\hat{Y}$ share the same origin $O \in Y, \hat{Y}$ for $O = 0 + 0i + 0j$; and

2. there is a one-to-one mapping of mathematical location elements between Y and $\hat{Y}$.

Second, for Santilli's geno-topic numbers [57, 58], we see that the right $\vec{y}_A \hat{\times}_> \vec{y}_B$ and left $\vec{y}_A \hat{\times}_< \vec{y}_B$ geno-multiplication operations may establish the two different sets $\hat{Y}_>$ and $\hat{Y}_<$ from $\hat{Y}$ with corresponding geno-units $\hat{r}_>$ and $\hat{r}_<$ to possibly identify *geno-triplex numbers* in accordance to [2]. Hence, Santilli's geno-multiplication applies to the triplex numbers of [52].

Third, for Santilli's hyper-topic numbers [57, 58], we clarify that the iso-multiplicative unit for $\vec{y}_A \hat{\times} \vec{y}_B$ is not limited to a unique value because it can comprise an ordered set of values (i.e. recall $\hat{r}_> = \{1, 1, 2, 3, 5, ...\}$ or $\hat{r}_> = \{2, \frac{4}{5}, 7, ...\}$) since the set is applicable to right and left [2]. Thus, it is evident that Santilli's hyper-multiplication may apply to the triplex numbers of [52] and possibly identify the pertinent *hyper-triplex numbers* in accordance to [2].

Fourth, for Santilli's iso-dual numbers [57, 58], the iso-multiplicative unit $\hat{r}$ may be any value except for zero, so Santilli's method [57, 58] may apply to the triplex numbers of [52] to possibly identify the relevant *iso-dual iso-triplex numbers* in accordance to [2].

And finally, at this point, these initial results of our preliminary assessment suggest the existence of geno-triplex, hyper-triplex, and iso-dual-triplex numbers, with direct application to *iso-fractals*, *geno-fractals*, *hyper-fractals*, and *iso-dual-fractals*—certainly, the iso-triplex numbers and iso-triplex space of eqs. (35–38) engage an iso-fractal version of the Mandelbulb Set in eq. (31). Therefore, these topics will be subject for future investigation beyond the limited scope of this paper.

3.2 Iso-2-sphere HR topology

Here, in the second phase of engagement, the primary objective is to apply Santilli's framework [2, 57, 58, 59, 60] to construct a preliminary explanation on how the geometry of $T^2 \subset Y$ in Inopin's 2-sphere HR topology [52, 56] can be reduced to the *one single geodesic* T^1, where $T^1 \subset T^2 \subset Y$, over the iso-triplex numbers.

First, given that the T^1 of eq. (16) and T^2 of eq. (33) are both triplex sub-spaces of Y, then Lemmas 1.1–1.2 apply to $T^1, T^2 \subset Y$ so they can be

iso-topically lifted to $\hat{T}^1, \hat{T}^2 \subset \hat{Y}$, respectively, where we define $\hat{T}^1$ as the *iso-1-sphere HR* and $\hat{T}^2$ as the *iso-2-sphere HR*. This enables us to apply eq. (38) to define the iso-topic liftings (and the inverses) for $T^1, T^2 \subset Y$ with some iso-unit $\hat{r}$ as

$$\begin{array}{rrcl} f(\hat{r}): & T^1 & \rightarrow & \hat{T}^1 \\ f^{-1}(\hat{r}): & \hat{T}^1 & \rightarrow & T^1 \end{array} \tag{39}$$

and

$$\begin{array}{rrcl} f(\hat{r}): & T^2 & \rightarrow & \hat{T}^2 \\ f^{-1}(\hat{r}): & \hat{T}^2 & \rightarrow & T^2, \end{array} \tag{40}$$

respectively. At this point, we identify:

- **Lemma 2.1**: An Inopin n-sphere HR T^n that is iso-metrically embedded in a triplex space Y, such that $n \in \{1, 2\}$, can be iso-topically lifted via the transition $f(\hat{r}): T^n \rightarrow \hat{T}^n$ (with iso-unit $\hat{r}$) to become an iso-n-sphere HR $\hat{T}^n$ when the superseding Y is simultaneously lifted to $\hat{Y}$, where T^n and $\hat{T}^n$ are locally iso-morphic.

- **Lemma 2.2**: An Inopin n-sphere HR T^n and its corresponding (iso-topically lifted) iso-n-sphere HR $\hat{T}^n$ are locally iso-morphic because:

  1. T^n and $\hat{T}^n$ share the same center $O \in Y, \hat{Y}$ for $O = 0 + 0i + 0j$;
  2. there is a one-to-one mapping of mathematical location elements between Y and $\hat{Y}$; and
  3. Y and $\hat{Y}$ are locally iso-morphic.

Now, the iso-morphic characterization of Y, T^1, and T^2 in Lemmas 1.1–1.2 and 2.1–2.2, which is one initial outcome of applying Santilli's iso-number framework [2, 57, 58, 59, 60], is *fundamentally* compliant with the fact that Inopin's 2-sphere HR topology can be equipped with various configurations of complex and triplex order parameter fields to encode topological deformations for a wide spectrum of physical features [52]. In other words, the order parameter field is a powerful tool that enables us to define topological quantities and functions for states and transitions associated with, for example, iso-morphism and/or stereo-graphic projections [52, 56]. Additionally, Santilli [57, 58] *proved* that the infinite set of all possible ellipsoids

arising from topologically-preserving deformations of a sphere in conventional Euclidean space over a conventional field are mapped to a corresponding Santilli iso-sphere in a Euclidean-Santillian iso-real space. More specifically, Santilli [57, 58] demonstrated that the mechanism for this is quite simple: the procedure consists of embedding all deformations $\hat{\kappa}\delta$ of Euclid's metric in the iso-unit $\hat{r} = \frac{1}{\hat{\kappa}} > 0$, such that $\delta = Diag.(1, 1, 1)$—any observer who looks at the resulting image of the ellipsoid under this map will see only *one* object: the perfect sphere with an amplitude-radius of $\hat{r}$. Therefore, given Lemmas 1.1–1.2 and 2.1–2.2, the infinite set of all possible ellipsoids arising from topologically-preserving deformations of $T^2 \subset Y$ are mapped to the corresponding $\hat{T}^2 \subset \hat{Y}$, where the positive-definite amplitude-radius is $r = \hat{r}$ and the corresponding curvature is $\kappa = \hat{\kappa} = \frac{1}{\hat{r}}$. To interpret this enlightening result in the case of, for example, Riemann geometry, we select a geodesic of Y with the 3x3 iso-metric $\hat{g}$ for Inopin's 2-sphere HR topology, then all infinitely possible Riemannian metrics G can be factorized in the form

$$G = \hat{\kappa} \times \hat{g}. \tag{41}$$

Thus, for T^2's great circle geodesic of T^1, the reformulation of Y in terms of $\hat{Y}$ with iso-unit $\hat{r}$ can be reduced to the analysis of only *one* circular element—namely Inopin's T^1 for $\hat{T}^1$—with a geodesic characterized by $\hat{g}$ because all infinitely possible circular elements in Y are obtained via (stereographic) projection! Immediately, we observe the resemblance of Santilli's *interior* and *exterior* dynamical systems [57, 58, 59, 60] to Y_- and Y_+, respectively, in terms of the iso-multiplicative unit $\hat{r}$ for the corresponding $\hat{Y}_-$ and $\hat{Y}_+$. Thus, we've defined an *iso-2-sphere HR topology* of $\hat{Y}$. At this point, we identify:

- **Lemma 3.1**: An Inopin 1-sphere HR $T^1 \subset Y$ (the great circle of the Inopin 2-sphere HR $T^2 \subset Y$) that is iso-metrically embedded in the Euclidean triplex space Y is one non-linear structure that satisfies the strict iso-curvature and iso-duality constraints of the iso-2-sphere HR topology, where $Y_- \subset Y$ and $Y_+ \subset Y$ correspond to Santilli-Inopin interior and exterior dynamical systems, respectively, such that the complete Y (with its said topological sub-spaces) are iso-morphic to their respective iso-topic liftings.

- **Lemma 3.2**: If a Euclidean triplex space Y that is iso-metrically equipped with the Inopin 2-sphere HR $T^2 \subset Y$ and its great circle, namely the Inopin 1-sphere HR $T^1 \subset Y$, is locally iso-morphic to its iso-topic lifting $\hat{Y}$ (iso-metrically equipped with the iso-2-sphere HR $\hat{T}^2$), then all forms of the iso-1-sphere HR $\hat{T}^1$ may be unified into one single iso-circular form on $\hat{Y}$, where $\hat{Y}_- \subset \hat{Y}$ and $\hat{Y}_+ \subset \hat{Y}$ are iso-dual because they are simultaneously dual to $\hat{T}^2$ for which $\hat{T}^1$ is the great circle.

The said geometric and non-linear unifications are parallel to that for Lie algebras because all possible, simple Lie algebras (compact and non-compact) of dimension d can be unified into one single *Lie-Santilli* iso-tope of dimension d (with the sole exclusion of exceptional algebra currently being investigated from other perspectives). Moreover, we achieve the capability of dealing with non-linear, non-local, and non-Hamiltonian systems at the Lie-Santilli level, which currently remains unattainable for the conventional Lie.

4 Conclusion

In this work, we started by briefly discussing the importance and development of number systems in terms of science. We touched on the five original number field axioms [2, 3], the abundance of fractals and chaos in nature, and acknowledged the significance of identifying a universal number classification system. Subsequently, we identified Santilli's four distinct data structure classes, namely the iso-numbers, the geno-numbers, the hyper-numbers, and the iso-dual numbers, [2, 57, 58, 59, 60], which are pertinent to an application assessment of the triplex numbers [52], triplex fractals, and Inopin's HR topology [52, 56].

Next, we conducted a preliminary upgrade to the triplex numbers and Euclidean triplex space of Inopin's 2-sphere HR topology [52] by engaging Santilli's four iso-number classes [2, 57, 58, 59, 60]. In doing so, we identified and defined the Santilli iso-triplex numbers to construct the Euclidean iso-triplex space for 3D iso-fractals, and subsequently demonstrated the existence of geno-triplex numbers, hyper-triplex numbers, and iso-dual-triplex numbers in a preparatory context. Afterwards, we provided a preliminary explanation on how Santilli's iso-mathematics [57, 58, 59, 60] apply to the

Inopin HR [52, 56] by defining the iso-2-sphere HR (with the iso-1-sphere HR as its great circle) to assemble the iso-2-sphere HR topology. Consequently, we stated an array of lemmas that aim to characterize these emerging mathematical structures.

In total, the resulting constructions of this venture are significant because they are cutting-edge, and therefore advance the frontiers of iso-mathematics to new realms of exploration and application. Hence, with the objective of further implementing these developments in the disciplines of science, technology, and engineering, we propose that a thorough and rigorous iso-mathematical investigation should be conducted along this research trajectory to challenge, upgrade, and generalize this emerging framework.

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DECELERATION PARAMETER Q(Z) AND EXAMINING IF A JOINT DM-DE MODEL IS FEASIZABLE, WITH APPLICATIONS TO "ATOMS OF SPACE-TIME" THERMODYNAMICS AND BBN

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Abstract

The case for a four dimensional graviton mass (non zero) influencing reacceleration of the universe in five dimensions is stated, with particular emphasis upon if five dimensional geometries as given below give us new physical insight as to cosmological evolution. One noticeable datum, that a calculated inflaton $\phi(t)$ may partly re-emerge after fading out in the aftermath of inflation. The inflaton may be a contributing factor to, with non zero graviton mass, in re acceleration of the universe a billion years ago. Many theorists assume that the inflaton is the source of entropy. The inflaton also may be the source of re acceleration of the universe, especially if the effects of a re emergent inflaton are in tandem with the appearance of macro effects of a small graviton mass, leading to a speed up of the rate of expansion of the universe one billion years ago, at red shift value of $Z \sim .423$. The key formula, for joint DM-DE shows up in terms of deceleration parameter, Q(z). The choice of the DM-DE eqn. mayeventually illuminate

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how early BBN may affect the formation of low levels of lithium for early star formation which we reference toward the end of this document. We also discuss what is necessary for not only proper BBN, but also to the implications for 'atoms' of space time congruent with relic GW production, i.e. the thermodynamics of emergent structure

Keywords: inflaton ,non zero graviton mass, emergent structure, BBN

1 Introduction

We will start with a first-principle introduction to detection of gravitational wave density using the definition given by Maggiore [19]

$$\Omega_{gw} \equiv \frac{\rho_{gw}}{\rho_c} \equiv \int_{f=0}^{f=\infty} d(\log f) \cdot \Omega_{gw}(f) \Rightarrow h_0^2 \Omega_{gw}(f) \cong 3.6 \cdot \left[\frac{n_f}{10^{37}}\right] \cdot \left(\frac{f}{1kHz}\right)^4 \quad (1)$$

where n_f is the frequency-based numerical count of gravitons per unit phase space. The author suggests that n_f may also depend upon the interaction of gravitons with neutrinos in plasma during early-universe nucleation, as modeled by M. Marklund*et al*[20]. Having said that, the question is, what sort of mechanism is appropriate for considering macro affects of gravitons, and the author thinks that he has one, i.e. reacceleration of the universe, as far as a function of graviton mass, i.e. what Beckwith [9] did was to make the following presentation. Assume Snyder geometry and look at use of the following inequality for a change in the HUP, [9]

$$\Delta x \geq \left[(1/\Delta p) + l_s^2 \cdot \Delta p\right] \equiv (1/\Delta p) - \alpha \cdot \Delta p \quad (2)$$

and that the mass of the graviton is partly due to the stretching alluded to by Fuller and Kishimoto,[13] a supposition the author [9] is investigating for a modification of a joint KK tower ofgravitons, as given by Maartens[17] for DM. Assume the stretching of early relic neutrinos that would lead to the KK tower of gravitons--for when $\alpha < 0$, is[6],

$$m_n(Graviton) = \frac{n}{L} + 10^{-65} \text{ grams} \quad (3)$$

Note that Rubakov[26] writes KK graviton representation as, after using the following normalization $\int \frac{dz}{a(z)} \cdot [h_m(z) \cdot h_{\tilde{m}}(z)] \equiv \delta(m - \tilde{m})$ where J_1, J_2, N_1, N_2 are different forms of Bessel functions, to obtain the KK graviton/ DM candidate representation along RS dSbrane world [26]

$$h_m(z) = \sqrt{m/k} \cdot \frac{J_1(m/k) \cdot N_2([m/k] \cdot \exp(k \cdot z)) - N_1(m/k) \cdot J_2([m/k] \cdot \exp(k \cdot z))}{\sqrt{[J_1(m/k)]^2 + [N_1(m/k)]^2}} \quad (4)$$

This (4) is for KK gravitons having a TeV magnitude mass $M_Z \sim k$ (i.e. for mass values at .5 TeV to above a TeV in value) on a negative tension RS brane. What would be useful would be managing to relate this KK graviton, which is moving with a speed proportional to H^{-1} with regards to the negative tension brane with $h \equiv h_m(z \to 0) = const \cdot \sqrt{\frac{m}{k}}$ as an initial starting value for the KK graviton mass, before the KK graviton, as a 'massive' graviton moves with velocity H^{-1} along the RS dSbrane. If so, and if $h \equiv h_m(z \to 0) = const \cdot \sqrt{\frac{m}{k}}$ represents an initial state, then one may relate the mass of the KK graviton, moving at high speed, with the initial rest mass of the graviton, which in four space in a rest mass

configuration would have a mass lower in value, i.e. of $m_{graviton}(4-Dim\ \ GR)\sim 10^{-48}eV$, as opposed to $M_X \sim M_{KK-Graviton}\sim .5\times 10^9 eV$. Whatever the range of the graviton mass, it may be a way to make sense of what was presented by Dubovsky et.al [11]. who argue for graviton mass using CMBR measurements, of $M_{KK-Graviton}\sim 10^{-20}eV$ Duboskyet. al. [11]results can be conflated with Alveset. al. [1] arguing that non zero graviton mass may lead to an acceleration of our present universe, in a manner usually conflated with DE, i.e. their graviton mass would be about $m_{graviton}(4-Dim\ \ GR)\sim 10^{-48}\times 10^{-5}eV\sim 10^{65}$ grams. Also assume that to calculate the deceleration, the following modification of the HUP is used: [2] $\Delta x \geq \left[(1/\Delta p)+l_s^2\cdot\Delta p\right]\equiv(1/\Delta p)-\alpha\cdot\Delta p$, where the LQG condition is $\alpha>0$, and brane worlds have, instead, $\alpha<0$[4]. Also (5) will be the starting point used for a KK tower version of (6) below. So from Maarten's [18]2005 paper,

$$\dot{a}^2=\left[\left(\frac{\tilde{\kappa}^2}{3}\left[\rho+\frac{\rho^2}{2\lambda}\right]\right)a^2+\frac{\Lambda\cdot a^2}{3}+\frac{m}{a^2}-K\right] \quad (5)$$

Maartens [18]also gives a 2nd Friedman equation, as

$$\dot{H}^2=\left[-\left(\frac{\tilde{\kappa}^2}{2}\cdot[p+\rho]\cdot\left[1+\frac{\rho^2}{\lambda}\right]\right)+\frac{\Lambda\cdot a^2}{3}-2\frac{m}{a^4}+\frac{K}{a^2}\right] \quad (6)$$

Also, if we are in the regime for which $\rho\cong -P$, for red shift values z between zero to 1.0-1.5 with exact equality, $\rho=-P$, for z between zero to **.5.** The net

effect will be to obtain, due to (6), and use $a \equiv [a_0 = 1]/(1+z)$. As given by Beckwith[3]

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \equiv -1 - \frac{\dot{H}}{H^2} = -1 + \frac{2}{1+\tilde{\kappa}^2[\rho/m]\cdot(1+z)^4 \cdot (1+\rho/2\lambda)} \approx -1 + \frac{2}{2+\delta(z)} \quad (7)$$

(6) assumes $\Lambda = 0 = K$, and the net effect is to obtain, a substitute for DE, by presenting how gravitons with a small mass done with $\Lambda \neq 0$, even if curvature **K** =0

2. Consequences of small graviton mass for reacceleration of the universe

In a revision of Alves*et. al*[1], Beckwith[9] used a higher-dimensional model of the brane world and Marsden[17] KK graviton towers. The density ρ of the brane world in the Friedman equation as used by Alves*et. al*[9] is use by Beckwith[9] for a non-zero graviton

$$\rho \equiv \rho_0 \cdot (1+z)^3 - \left[\frac{m_g \cdot (c=1)^6}{8\pi G(\hbar = 1)^2}\right] \cdot \left(\frac{1}{14\cdot(1+z)^3} + \frac{2}{5\cdot(1+z)^2} - \frac{1}{2}\right) \quad (8)$$

I.e.(6) above is making a joint DM and DE model, with all of . (6) being for KK gravitons and DM, and 10^{-65} grams being a 4 dimensional DE. (5) is part of a KK graviton presentation of DM/ DE dynamics. Beckwith [9] found at $z \sim .4$, a billion years ago, that acceleration of the universe increased, as shown in Fig. 1.

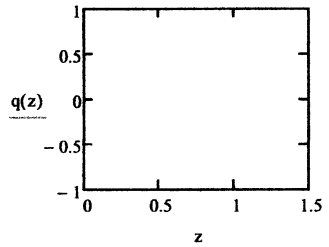

Fig. 1: Reacceleration of the universe based on Beckwith [9]**(note that q < 0 if z <.423)**

3. Suggesting a non standard way to accommodate small graviton mass in 4 D

If one is adding the small mass of $m_n(Graviton) = \frac{n}{L} + 10^{-65}$ grams [9], with $m_0(Graviton) \approx 10^{-65}$ grams, then the problem being worked with is a source term problem of the form given by Peskins[24] as of the type

$$\psi_n(x) \equiv \int d^3p \cdot \frac{1}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_P}} \cdot \left\{ \left(a_P + \frac{i}{\sqrt{2E_P}} \cdot FT(m_0(graviton)) \right) \exp(-ipx) + H.C. \right\} \quad (8a)$$

This is, using the language V.A. Rubakov[26] put up equivalent to[9],

$$\psi_m(x) \approx h_m(x) + \int d^3p \cdot \frac{1}{(2\pi)^3} \cdot \left(\frac{1}{\sqrt{2E_P}} \right)^2 \cdot \{ (i \cdot FT(m_0(graviton))) \exp(-ipx) + H.C. \} \quad (9)$$

If $m_0(graviton)$ is a constant, then the expression (9) has delta functions. This is the field theoretic identification. Another way is to consider an instanton-anti instanton treatment of individual gravitons, and to first start with the supposed stretch out of gravitons to enormous lengths. Assuming $m_0(Graviton) \approx 10^{-65}$ grams for gravitons in 4 dimensions, the supposition by Bashinsky[2] and Beckwith[9] is that density fluctuations are influenced by a modification of cosmological density ρ in the Friedmann equations by the proportionality factor given by Bashinsky [2], $\left|1-5\cdot(\rho_{neutrino}/\rho)+\vartheta\left[\rho_{neutrino}/\rho\right]^2\right)\right|$ This proportionality factor for ρ as showing up in the Friedmann equations should be taken as an extension of results from Marklund*et. al*[20] , due to graviton-neutrino interactions as proposed by Marklund*et al*[20], where neutrinos interact with plasmons andplasmons interact with gravitons. Thereby implying neutrino-graviton interactions Also, graviton wavelengths have the same order of magnitude of neutrinos. Note, from Valev [31],

$$m_{graviton}\Big|_{RELATIVISTIC} < 4.4\times 10^{-22} h^{-1} eV/c^2$$
$$\Leftrightarrow \lambda_{graviton} \equiv \frac{\hbar}{m_{graviton}\cdot c} < 2.8\times 10^{-8}\, meters \qquad (10)$$

Extending M. Marklund*et al* [20]and Valev[31], some gravitons may become larger[32], i.e.

$$\lambda_{graviton} \equiv \frac{\hbar}{m_{graviton}\cdot c} < 10^{4}\, meters \text{ or larger} \qquad (11)$$

A way to accommodate this wave length has been suggested by Beckwith,[3] as to an instanton-anti instanton packaging of gravitons, was to start with an analogy between Giovannini,[14] from a least action version of the Einstein – Hilbert action for 'quadratic' theories of gravity involving Euler- Gauss-Bonnet. Then Giovannini's [14]equation 6 corresponds to

$$\phi = \tilde{\nu} + \arctan\left((bw)^{\nu}\right) \quad (12)$$

Givannini [14] represents of (12) as a kink, and makes references to an anti-kink solution, in Fig. 1 in Givannini [14].Furthermore the similarity between (12) and $\phi_{+}(z,\tau) = 4 \cdot \arctan\left(\exp\left\{ \frac{z+\beta \cdot \tau}{\sqrt{1-\beta^2}} \right\} \right)$ in Beckwith's [4,9] treatment with regards to density wave physics instantons is obvious. If $\arctan\left((bw)^{\nu}\right)$ is part of representing a graviton as a kink-anti-kink combination, arising from a 5 dimensional line element, **[14]**

$$dS^2 = a(w) \cdot \left[\eta_{uv} dx^u dx^v - dw^2 \right] \quad (13)$$

Then, noting as Beckwith[9] mentioned, there is the possibility of using t'Hoofts [28,29] classical embedding of "deterministic quantum mechanics" as a way to embed a nearly four dimensional graviton as having almost zero mass, in a larger non linear theory.

4. What if an inflaton partly re-emerges in space-time dynamics? At z ~ . 423?

Padmanabhan[22] has written up how the 2nd Friedman equation as of (5), which for **z ~ . 423** may be simplified to readas

$$\dot{H}^2 \cong \left[-2\frac{m}{a^4}\right] \qquad (14)$$

would lead to an inflaton value of , when put in, for scale factor behavior as given by $a(t) \propto t^{\lambda}, \lambda = (1/2) - \varepsilon^{+}, 0 \le \varepsilon^{+} << 1$, of, for the inflaton[18] and inflation of

$$\phi(t) = \int dt \cdot \sqrt{-\frac{\dot{H}}{4\pi G}} \qquad (15)$$

Assuming a decline of $a(t) \propto t^{\lambda}, \lambda = (1/2) - \varepsilon^{+}, 0 \le \varepsilon^{+} << 1$, (15) yields

$$\phi(t) \sim \sqrt{\frac{2m}{4\pi G}} \cdot \left[2\varepsilon^{+}\right] \cdot t^{2 \cdot \varepsilon^{+}} \qquad (16)$$

As the scale factor of $a(t) \propto t^{\lambda}, \lambda = (1/2) - \varepsilon^{+}, 0 \le \varepsilon^{+} << 1$ had time of the value of roughly $a(t) \propto t^{\lambda}, \lambda = (1/2) - \varepsilon^{+}, 0 \le \varepsilon^{+} << 1$ have a power law relationship drop below $a(t) \propto t^{1/2}$, the inflaton took (16) 's value which may have been a

factor as to the increase in the rate of acceleration, as noted by the q factor , given in Fig. 1. Note that there have been analytical work projects relating the inflaton, and its behavior to entropy via noting that inflation stopped when the inflaton field settled down into a lower lower energy state. The way to relate an energy state to the inflatonis , if $a(t) = a_0 t^{\lambda}$, then in the early universe, one has a potential energy term of[23]

$$V(\phi) = V_0 \cdot \exp\left[-\sqrt{\frac{16\pi G}{\lambda}} \cdot \phi(t) \right] \quad (17)$$

A situation where both $\lambda = (1/2) - \varepsilon^{+}$ grows smaller, and, temporarily, $\phi(t)$takes on (16)'s value, even if the time value gets large, and also, if acceleration of the cosmic expansion is taken into account, then there is infusion of energy by an amount dV. The entropy dS≈dV/T, will lead, if there is an increase in V, as given by (17) a situation where there is an effective increase in entropy. If there is, as will be related to later, in page eight, circumstances, where $S \approx N =$ number of graviton states[9,22]as will be derived in (27), then at least in higher dimensions, we have an argument that the re emergence of an inflaton, with a corresponding reduction of (17) in magnitude may be part of gravitons playing a role in the re acceleration of the universe.

5. Other than five dimensions for cosmology? Problems which need resolutions

If a way to obtain a graviton mass in four dimensions is done which fits in with the as given higher 5 dimensions specified by a slight modification of brane theory, or Maarten's cosmological evolution[9,18] equations, what benefits could this approach accrue for other outstanding problems in cosmology ? Beckwith[3]claims that a re do of the Friedmann equations would result in

deceleration parameter q(z) similar to Fig. 1 above. Snyder geometry for the four dimensional case with would specify Friedmann equations along the lines of $\alpha > 0$ in (2) above. If one follows $\alpha < 0$, then the Friedmann equations appear as giving details to the following equation [9,27]

$$\Im = -\frac{1}{2}\int d^4 x h^0_{uv} \cdot T^{uv} \sim L^2 \approx \delta^+ \geq 0 \qquad (18)$$

The construction done from sections 1 to 3 are for $\alpha < 0$. When $\alpha > 0$, the claim is that almost all the complexity is removed $\alpha > 0$, and what is left is a Taveras [27]treatment of the Friedmann equations, where he obtains, to first order, if ρ is a scalar field density,

$$\left(\frac{\dot{a}}{a}\right)^2 = [\kappa/3]\cdot\rho \qquad (19)$$

and

$$\left(\frac{\ddot{a}}{a}\right) = -[2\cdot k/3]\cdot\rho \qquad (20)$$

The interpretation of ρ as a scalar field density [27]and if one does as Alves et al [1]uses (7) above. We need to interpret the role of ρ. In the LQG version by [6], (20) may be rewritten as follows: If conjugate momentum is in many cases, "almost" or actually a constant, using $\dot{\phi} = -[\hbar/i]\cdot[\partial/\partial p_\phi]$

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv [\kappa/6]\cdot\left[p_\phi^2/a^6\right] \tag{21}$$

Beckwith[9] claims that the deceleration parameter q (z) incorporating (19), (20) and. (21) should give much the same behavior as Fig. 1 above. If so, then if one is differentiating between four and five dimensions by what is gained, in cosmology, one needs having it done via other criteria. The following is a real problem. As given by Maggiore [19], the massless equation of the graviton evolution equation takes the form

$$\partial_\mu \partial^\varpi h_{\mu\nu} = \sqrt{32\pi G}\cdot\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T_\mu^\mu\right) \tag{22}$$

When $m_{graviton} \neq 0$, the above becomes[19]

$$\left(\partial_\mu \partial^\varpi - m_{graviton}\right)\cdot h_{\mu\nu} = \left[\sqrt{32\pi G} + \delta^+\right]\cdot\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T_\mu^\mu + \frac{\partial_\mu\partial_\nu T_\mu^\mu}{3m_{graviton}}\right) \tag{23}$$

The mismatch between these two equations, when $m_{graviton} \to 0$, is due to $m_{graviton}h_\mu^\mu \neq 0$ as $m_{graviton} \to 0$, which is due to setting a value of $m_{graviton}\cdot h_\mu^\mu = -\left[\sqrt{32\pi G} + \delta^+\right]\cdot T_\mu^\mu$ The semi classical method by t'Hooft [28,29] , using (12) is the solution. We generalize to higher dimensions the following diagram as given by Beckwith[9] . Use an instanton- anti instanton structure, and t'Hooft [27,28]

equivalence classes along the lines of (24) below with equivalence class structure in the below wave functional to be set by a family of admissible values [9] $\phi_0(x)$

$$\Psi_{i,f}\left[\phi(\mathbf{x})\right]\Big|_{\phi=\phi_{ci,cf}} = c_{i,f}\cdot\exp\left\{-\int d\mathbf{x}\,\alpha\left[\phi_{Ci,f}(\mathbf{x})-\phi_0(\mathbf{x})\right]^2\right\} \qquad (24)$$

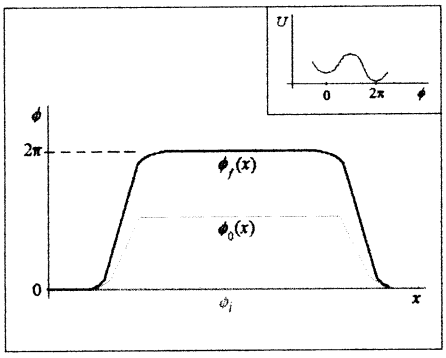

Fig. 2: The pop up effects of an intanton-anti-instanton in Euclidian space [9,28,29]

6. Tie in of Eq. (16) with structural issues of low lithium in early stars

What we are doing is to have (16) and (17) form an upper bound as to initial energy , making the identification of a net energy $E \propto \hbar\omega \propto V(\phi)$ at the onset of inflation with , when doing it $\omega \sim \omega_F$ which will serve as an upper bound to the expression below as part of the inequality so discussed for the formation of neutrino mass. Our discussion closely parallels what was introduced by Beckwith, in Erice [6] Note

Marklund, Brodin, and. Shukla [20] have estimated neutrino mass as $m_\nu^2 = -g_{\alpha\ \beta} p^\alpha p^\beta$ where m_ν is neutrino mass, $g_{\alpha\beta}$ is for a metric, and p^α is fourmomentum. If space becomes abruptly flat at the onset of inflation, for a neutrino mass, as $\bar{L} \neq 0$ approaches zero, $g_{\alpha\beta}$ approaches $g_{\alpha\alpha}$, i.e. leading to flat space, then by M. Marklund et al [20] there exits, assuming k^α is for a four space wave number, the inequality

$$\omega_F^2 > \left(g_{\alpha\alpha}/|g_{00}|\right)\cdot\left[k^\alpha\right]^2 + 2\omega_F\left(g_{00}/|g_{00}|\right)k^0 \qquad (25)$$

It is suggested that neutrino-graviton interactions would allow a researcher to input values of k^α, k^0, $g_{\alpha\alpha}$, and g_{00} when (25) is true, based on that the neutrino has approximately $10^{28} - 10^{29}$ the effective mass of a graviton. As seen in (10),

$m_{graviton} \leq 4.4\times10^{-22} h^{-1} eV/c^2 \Leftrightarrow \lambda_{graviton} \sim 2.8\times10^{15} meters$ versus having

$$m_{neutrino-relic-condt} \leq .5\times10^{-1} h^{-1} eV/c^2$$
$$\Leftrightarrow \lambda_{neutrino-relic-condt.} \equiv \frac{\hbar}{m_n \cdot c} \sim 2.8\times10^{-8} meters \qquad (26)$$

I.e., for non-relativistic conditions, the contribution of the neutrino is $10^{22} - 10^{23}$ times larger than that from a graviton, . So for a non-relativistic graviton $\mu/M_{Planck} \sim \bar{L} \propto \frac{c-v}{c} \Leftrightarrow \frac{p^0}{\mu} \leq 1$. Once we specify that it is likely that graviton-neutrino wave mixing took place as $\bar{L} \to 0$, we can consider entropy

contributions in the time neutrinos interacted with gravitons to perturbations on DM which may influence BBN[6].

7. How DM would be influenced by gravitons

The interrelationship of structure of the profile of a DM cluster , with any perturbations DM density profile [21]

$$\delta \equiv -\left[\frac{3}{2}\cdot\Omega_m\cdot H^2\right]^{-1}\cdot\nabla^2\Phi \qquad (27)$$

As told to the author by Matarre [21] , in July , 2009, in Como Italy, the gravitational potential has , perturbatively speaking an additional term f_{NL} added to variations in the gravitational potential term which Matarre gave as[21]

$$\Phi \equiv \Phi_L + f_{NL}\cdot\left[\Phi_L^2 - \left\langle\Phi_L^2\right\rangle\right] + g_{NL}\cdot\Phi_L^3 \qquad (28)$$

It is suggested that the function f_{NL} is largely due to entropy variations, some of which occurred during relic GW/graviton production. Here the expression f_{NL} = variations from gaussianity. Furthermore, Φ_L is a linear Gaussian potential, and the overall gravitational potential is altered by inputs from f_{NL}. Note that neutrinos flavor physics oscillations are not very important in terms of f_{NL}, as specified in conversations. Beckwith had in September 23, 2009 in Erice with

Georg Raffert[25] . Which leads to emphasizing the role of entropy processes due to graviton-neutrino physics, as $\overline{L} \to 0$

as written up by Beckwith[21]

8 1st part of .Conclusion

The real start to this investigation is to explain how, and why the star HE0107-5240 could form with so little lithium in the first place. As stated by Fuller et al[20], neutrinos could interact with DM potential wells in ways Beckwith thinks could influence deviations from standard galaxy hierarchy formation models which will also have a counter part in deviations in the BBN nucleosynthesis of light elements,by examining the role of temperature fluctuations modeled on (29) below ,leading to fluctuations affecting BBN element rarity [21].

$$(\delta T/T) \cong (1/3) \cdot \left[\Phi_L + \tilde{f}_{NL} \cdot \left(\Phi_L^2 - \langle \Phi_L \rangle^2\right)\right] \quad (29)$$

While (29) above would have its maximum impact for regions as of about red shift $Z \sim 1.5 - 2.0$, the impact of (29) would be as of red shifts $Z \sim 1000 - 1100$, with the corresponding $\tilde{f}_{NL}$ influenced by Bashinsky's[2] neutrino – gravition damping as stated by the coefficient of density fluctuation modified by $\left[1 - 5 \cdot (\rho_{neutrino}/\rho) + \vartheta\left[\rho_{neutrino}/\rho\right]^2\right)\Big]$. Note that $\tilde{f}_{NL}$ would be larger than f_{NL} of Eq. (28) and would be dominated by neutrino-graviton interactions, whereas f_{NL} would be dominated by graviton generated entropy, with neutrinos at $Z \sim 2.0$ hitting DM directly. We submit that a graviton with a small rest mass may be more amendable to such interaction with neutrinos, and that in addition (27), (28) and (29) may influence and affect structure formation as seen by the following diagram in figure 1. Note that this is assuming that early universe interactions which we are talking about eventually play out and reach, with the re acceleration of the universe, as outlined in the 1st half of our document to also be indirectly responsible for the famous "halo merging tree

diagram we call Fig 3 below.At or about when $k \geq k_{equilibrium} \equiv \tau^{-1}_{equilibrium} \sim 10^{-2} Mpc^{-1}$ begins to delineate the neutrino-GW interaction becoming a significant damping impact upon each other, one would be seeing variations from the usual structure formation, as given by the following diagram.[33]

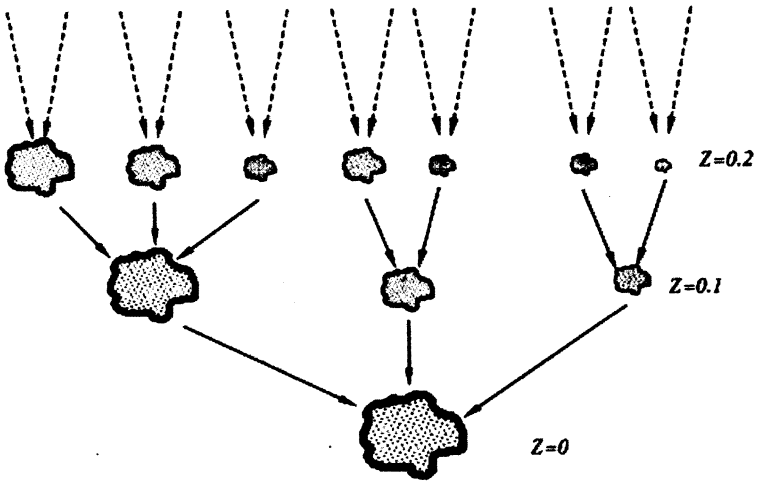

Figure 1. A schematic representation of a halo merging history 'tree'.

Figure 3 how we obtain 'bottom up' development of galactic super structure which ducplicates a diagram given in reference[33]

We should keep in mind that the following holds, i.e. for flat space. That one will have Figure 3 in both flat and in curved space. Also note that, M. Marklund, G. Brodin, and P.K. Shukla[20] posted their own version of not only neutrino mass,

as given by $m_\nu^2 = -g_{\alpha\ \beta} p^\alpha p^\beta$, where the overall mass is set by Note, here, that the potential for where the frequency comes from is, here , is $U = \hbar \cdot \omega_F$, and , according to Eberleand Ringwald et al.[12], may have lightest relic neutrino masses of the order of

$$m_{relic-neutrino} \propto .1eV/c^2 \qquad (30)$$

as opposed to, as given by D, Valev[31]

$$m_{graviton} \leq 2\times 10^{-29} \breve{h}^{-1} eV/c^2 \qquad (31)$$

Where $\breve{h} \approx .65$, is a dimensionless Hubble constant, Very roughly put, for relic early universe conditions, one may be seeing that the neutrino has $10^{28} - 10^{29}$ the effective mass than a graviton. Furthermore, for a neutrino we have [6]

$$\lambda_k \approx \frac{hc}{E_k} + \frac{hm_{vk}^2 c^5}{2E_k^3} \qquad (32)$$

This will tie in directly with a neutrino mass limit we state as [6,20]

$$m_\nu^2 = -g_{\alpha\ \beta} p^\alpha p^\beta \equiv \left[\hbar \cdot \sqrt{|g_{00}| \cdot \omega_F^2 - g_{\alpha\beta} k^\alpha k^\beta - 2\omega_F g_{0\alpha} k^\alpha}\right]^2 \quad ..(33)$$

If , as if often expected in inflation, space becomes abruptly flat at the onset of inflation, then for a neutrino mass, as the $\overline{L} \xrightarrow[approach-to-standard-model-physics]{} 0$ will then lead to the following inequality[6,20].

$$m_\nu^2 \equiv \left[\hbar \cdot \sqrt{|g_{00}| \cdot \omega_F^2 - g_{\alpha\beta} k^\alpha k^\beta - 2\omega_F g_{0\alpha} k^\alpha}\right]^2$$

$$\xrightarrow[Flat-Space]{} \left[\hbar \cdot \sqrt{|g_{00}| \cdot \omega_F^2 - g_{\alpha\alpha}\left[k^\alpha\right]^2 - 2\omega_F g_{00} k^0}\right]^2 > 0 \qquad (34)$$

$$\Leftrightarrow |g_{00}| \cdot \omega_F^2 > g_{\alpha\alpha}\left[k^\alpha\right]^2 + 2\omega_F g_{00} k^0 \Rightarrow |g_{00}| \cdot \omega_F^2 > g_{\alpha\alpha}\left[k^\alpha\right]^2 + 2\omega_F g_{00} k^0$$

Now, how would variation from the above " halo Merging history tree', partly due to the modulation , via entropy, of DM structure formation, due to GW/gravitons affecting DM profile affect the concentration for lithium in stars, and perhaps lead to the famous 'lithium problem" being resolved ? We are investigating it. But we do think that having a graviton with mass is affecting the particulars of the 'halo mixing tree' diagram[30].

9. 2nd part of Conclusion. Examining information exchange between different universes?

Beckwith[9] has concluded that the only way to give an advantage to higher dimensions as far as cosmology would be to look at if a fifth dimension may present a way of actual information exchange to give the following parameter input from a prior to a present universe, i.e. the fine structure constant, as given by[9]

$$\tilde{\alpha} \equiv e^2/\hbar \cdot c \equiv \frac{e^2}{d} \times \frac{\lambda}{hc} \qquad (35)$$

The wave length as may be chosen to do such an information exchange would be part of a graviton as being part of an information counting algorithm as can be

put below, namely: Argue that when taking the log, that the 1/N term drops out. As used by Ng [22]

$$Z_N \sim (1/N!)\cdot\left(V/\lambda^3\right)^N \qquad (36)$$

This, according to Ng [22], leads to entropy of the limiting value of, if $S = (\log[Z_N])$ will be modified by having the following done, namely after his use of quantum infinite statistics, as commented upon by Beckwith [9]

$$S \approx N\cdot\left(\log\left[V/\lambda^3\right]+5/2\right)\approx N \qquad (37)'$$

Eventually, the author hopes to put on a sound foundation what 'tHooft [28,29] is doing with respect to t'Hooft [28,29] deterministic quantum mechanics and equivalence classes embedding quantum particle structures.. Doing so will answer the questions Kay[15] raised about particle creation, and the limitations of the particle concept in curved and flat space, i.e. the global hyperbolic space time which is flat everywhere expect in a localized "bump" of curvature. Furthermore, making a count of gravitons with $S \approx N \sim 10^{20}$ gravitons [9,28,29], with $I = S_{total}/k_B \ln 2 = [\#operations]^{3/4} \sim 10^{20}$ as implying at least one operation per unit graviton, with gravitons being one unit of information, per produced graviton[3]. What the author, Beckwith, sees is that since instanton- anti instanton pairs do not have to travel slowly, as has been proved by authors in the 1980s, that gravitons if nucleated in a fashion as indicated by Fig. 2, may be able to answer the following. The stretch-out of a graviton wave, greater than the size of the solar system, gives, an upper limit of a graviton mass due to wave length $\lambda_{graviton} > 300\cdot h_0 kpc \Leftrightarrow m_{graviton} < 2\times 10^{-29} h_0^{-1} eV$. I. e. stretched graviton wave, at ultra-low frequency, may lead to a low mass limit. However, more

careful limits due to experimental searches, as presented by Buonanno[10] have narrowed the upper limit to $10^{-20} h_0^{-1} eV$. An instanton – anti instanton structure to the graviton, if confirmed, plus experimental confirmation of mass, plus perhaps $n \sim 10^{20}$ gravitons $\approx 10^{20}$ entropy counts (23) implies up to $\approx 10^{27}$ operations. If so, there is a one-to-one relationship between an operation and a bit of information, so a graviton has at least one bit of information. And that may be enough to determine the conditions needed to determine if (21) gives information and structure from a prior universe to our present cosmos. Finally, the datum referred to in (14) to . (17) as combined with $S \approx N$ as referenced on pages 5 and 6 as a way to relate the graviton count with entropy may be a way to make inter connection between the inflaton picture of entropy generation and entropy connected/ generated with a numerical count of gravitons. This datum needs experimental confirmation and is important to astro physics linkage of DE with DM, in the future. (14) to. (17) if confirmed for Z ~ . 423 may prove that higher dimensions are necessary for cosmology.

10. 3rd and final point to conclusions , the need to find out the border of the introduction of where Quantum gravity emerges from a prior 'analog' structure may, if tied into questions of graviton mass determine if multiple universes are possible/ feasible. As well as extending a new "atoms of space time" of GR 'thermodynamics' to the early universe.

Beckwith[8], in his FQXi document outlined a procedure where a graviton with mass may be indicative of the existence of multiple universes co existing. The details of the mapping of that multiple universe picture involve a transition from an analog physics (discrete, i.e. classical world picture) to one where octonian gravity is formed ,i.e. a quantum picture as a pre cursor to quantum gravity. The

existence of a small mass may mean the extension of quantum physics to a larger embedding/ extension of quantum physics. Furthermore, keep in mind that tandem to that step of semi classical embedding of a graviton, that eventually we want to make explicit an idea by, T. Padmanabhan in DICE 2010 , as to finding "atoms of space time" permitting a thermodynamic treatment of emergent structure similar to Gibbs treatment of statistical physics[32]. I.e. for finding out if the following is possible, ie.can an ensemble of gravitons, be used to construct an 'atom' of space time congruent with relic GW. That is our ultimate end, as to our research. That would make our inquiry of the nature of gravitons most worthwhile. This idea was presented at DICE 2010, and we would like to refine it in our future research work. This would be in tandem of adapting the Kiefer, Polarski, and Starobinsky[14] presentation of the evolution of relic entropy via the evolution of phase spaces, with Γ/Γ_0 being the ratio of 'final (future)' / 'initial' phase space volume, for k modes of secondary GW background. From "atoms of space time" treatment of early universe space time geometry according to

$$S(k) = \ln \frac{\Gamma}{\Gamma_0} \tag{38}$$

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